6

The Russian school

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The towering figure of Kolmogorov and his very productive school is what was perceived in the twentieth century as the Russian school of turbulence. However, important Russian contributions neither start nor end with that school.

6.1 Physicist and pilot

> ...the bombs were falling almost the way the theory predicts. To have conclusive proof of the theory I'm going to fly again in a few days.
>
> A.A. Friedman, letter to V.A. Steklov, 1915

What seems to be the first major Russian contribution to the turbulence theory was made by Alexander Alexandrovich Friedman, famous for his work on non-stationary relativistic cosmology, which has revolutionized our view of the Universe. Friedman's biography reads like an adventure novel. Alexander Friedman was born in 1888 to a well-known St. Petersburg artistic family (Frenkel, 1988). His father, a ballet dancer and a composer, descended from a baptized Jew who had been given full civil rights after serving 25 years in the army (a so-called cantonist). His mother, also a conservatory graduate, was a daughter of the conductor of the Royal Mariinsky Theater. His parents divorced in 1897, their son staying with the father and becoming reconciled with his mother only after the 1917 revolution. While attending St. Petersburg's second gymnasium (the oldest in the city) Friedman befriended a fellow student Yakov Tamarkin, who later became a famous American mathematician and with whom he wrote their first scientific works (on number theory, received positively by David Hilbert). In 1906, Friedman and Tamarkin were admitted to the mathematical section of the Department of Physics and Mathematics of Petersburg University where they were strongly influenced by the

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great mathematician V.A. Steklov who taught them partial differential equations and regularly invited them to his home (with another fellow student V.I. Smirnov who later wrote the well-known *Course of Mathematics*, the first volume with Tamarkin). As his second, informal, teacher Alexander always mentioned Paul Ehrenfest who was in St. Petersburg in 1907–1912 and later corresponded with Friedman. Friedman and Tamarkin were among the few mathematicians invited to attend the regular seminar on theoretical physics in Ehrenfest's apartment. Apparently, Ehrenfest triggered Friedman's interest in physics and relativity, at first special and then general. During his graduate studies, Alexander Friedman worked on different mathematical subjects related to a wide set of natural and practical phenomena (among them on potential flow, corresponding with Joukovsky, who was in Moscow). Yet after getting his MSc degree, Alexander Friedman was firmly set to work on hydrodynamics and found employment in the Central Geophysical Laboratory. There, the former pure mathematician turned into a physicist, not only doing theory but also eagerly participating in atmospheric experiments, setting the measurements and flying on balloons. It is then less surprising to find Friedman flying a plane during World War I, when he was three times decorated for bravery. He flew bombing and reconnaissance raids, calculated the first bombardment tables, organized the first Russian air reconnaissance service and the factory of navigational devices (in Moscow, with Joukovsky's support), all the while publishing scientific papers on hydrodynamics and atmospheric physics. After the war ended in 1918, Alexander Alexandrovich was given a professorial position at Perm University (established in 1916 as a branch of St. Petersburg University), which boasted at that time Tamarkin, Besikovich and Vinogradov among the faculty. In 1920 Friedman returned to St. Petersburg. Steklov got him a junior position at the University (where George Gamov learnt relativity from him). Soon Friedman was teaching in the Polytechnic as well, where L.G. Loitsyansky was one of his students. In 1922 Friedman published his famous work *On the curvature of space* where the non-stationary Universe was born (Friedman, 1922). The conceptual novelty of this work is that it posed the task of describing the evolution of the Universe, not only its structure. The next year saw the dramatic exchange with Einstein, who at first published the paper that claimed that Friedman's work contained an error. Instead of public polemics, Friedman sent a personal letter to Einstein where he elaborated on the details of his derivations. After that, Einstein published the second paper admitting that the error was his. In 1924 Friedman published his work, described below, that laid down the foundations of the statistical theory of turbulence structure. In 1925 he made a record-breaking balloon flight to the height of 7400 meters to study atmospheric vortices and make medical self-observations. His personal life was quite turbulent at that time too: he was tearing himself between two women, a devoted wife since 1913 and another one pregnant with his child ("I do not have enough willpower at the moment to commit suicide" he wrote in a letter to the mother of his future son). On his way back from summer vacations by train in the Crimea, Alexander Friedman bought a nice-looking pear at a Ukrainian train station, did not wash it before eating and died from typhus two weeks later.

Friedman's work on turbulence theory was done in conjunction with his student Keller and was based on the works of Reynolds and Richardson, both cited extensively in Friedman and Keller (1925). Recall that Richardson derived the equations for the mean values which contained the averages of nonlinear terms that characterize turbulent fluctuations. Friedman and Keller cite Richardson's remark that such averaging would work only in the case of a so-called time separation when fast irregular motions are imposed on a slow-changing flow, so that the temporal window of averaging is in between the fast and slow timescales. For the first time, they then formulated the goal of writing down a closed set of equations for which an initial value problem for turbulent flow can be posed and solved. The evolutionary (then revolutionary) approach of Friedman to the description of the small-scale structure of turbulence parallels his approach to the description of the large-scale structure of the Universe. Achieving closure in the description of turbulence is nontrivial since the hydrodynamic equations are nonlinear. Indeed, if $\mathbf{v}$ is the velocity of the fluid, then Newton's second law gives the acceleration of the fluid particle:

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = \text{force per unit mass.} \tag{6.1}$$

Whatever the forces, the acceleration already contains the second (inertial) term, which makes the equation nonlinear. Averaging the fluid dynamical equations, one expresses the time derivative of the mean velocity, $\partial\langle\mathbf{v}\rangle/\partial t$, via the quadratic mean $\langle(\mathbf{v}\nabla)\mathbf{v}\rangle$. Friedman and Keller realized that meaningful closure can only be achieved by introducing correlation functions between different points in space and different moments in time. Their approach was intended for the description of turbulence superimposed on a non-uniform mean flow. Writing the equation for the two-point function $\partial\langle\mathbf{v}_1\mathbf{v}_2\rangle/\partial t$, they then derived the closed system of equations by decoupling the third moment via the second moment and the mean: $\langle v_1^i v_2^j v_2^k\rangle = \langle v_{1i}\rangle\langle v_{2j}v_{2k}\rangle + \cdots$ (Friedman and Keller, 1925). It is interesting that Friedman called the correlation functions "moments of conservation" (Erhaltungsmomenten) as they express "the tendency to preserve deviations from the mean values" in a curious resemblance to the modern approach based on martingales or zero modes. The work was presented at the

First International Congress on Applied Mechanics in Delft in 1925. During the discussion after Friedman's talk he made it clear that he was aware that the approximation is crude and that time averages are not well-defined. He stressed that his goal was pragmatic (predictive meteorology) and that only a consistent theory of turbulence can pave the way for dynamical meteorology: "Instruments give us mean values while hydrodynamic equations are applied to the values at a given moment". The introduction of correlation functions was thus the main contribution to turbulence theory made by Alexander Friedman, a great physicist and a pilot.

One year after Friedman's death, the seminal paper of Richardson on atmospheric diffusion appeared. I cannot resist imagining what would have happened if Friedman saw this paper and made a natural next step: to incorporate the idea of cascade and the scaling law of Richardson's diffusion into the Friedman–Keller formalism of correlation functions and to realize that the third moment of velocity fluctuations, which they neglected, is crucial for the description of the turbulence structure. As it happened, this was done 15 years later by another great Russian scientist, mathematician Andrei Nikolaevich Kolmogorov.

6.2 Mathematician

At any moment, there exists a narrow layer between trivial and impossible where mathematical discoveries are made. Therefore, an applied problem is either solved trivially or not solved at all. It is an altogether different story if an applied problem is found to fit (or made to fit!) the new formalism interesting for a mathematician.

A.N. Kolmogorov, diary, 1943

Russians managed to continue, well into the twentieth century, the tradition of great mathematicians doing physics.

Andrei Kolmogorov was born in 1903. His parents weren't married. The mother, Maria Kolmogorova, died at birth. The boy was named according to her wish after Andrei Bolkonski, the protagonist from the novel *War and Peace* by Lev Tolstoy. Andrei was adopted by his aunt, Vera Kolmogorova, and grew up in the estate of his grandfather, district marshal of nobility, near Yaroslavl. The father, agronomist Nikolai Kataev, took no part in his son's upbringing: he perished in 1919, fighting in the Civil War. Vera and Andrei relocated to Moscow in 1910. In 1920, Andrei graduated from the Madame

Repman gymnasium (cheap but very good) and was admitted to Moscow University, with which he remained associated for the rest of his life. In a few months, he passed all the first-year exams and was transferred to the second year which “gave the right to 16 kg of bread and 1 kg of butter a month – full material prosperity by the standards of the day” (Kolmogorov, 2001). His thesis adviser was Nikolai Luzin who ran the famous research group ‘Luzitania’. Apart from him, Kolmogorov was influenced by D. Egorov, V. Stepanov, M. Suslin, P. Urysohn and P. Aleksandrov, with whom Kolmogorov was close until the end of his life, sharing a small cottage in Komarovka village where they regularly invited colleagues and students, who described the unforgettable atmosphere of science, art, sport and friendship (Shiryaev, 2006). Kolmogorov completed his doctorate in 1929. In 1931, following a radical restructuring of the Moscow mathematical community, he was elected a professor. He spent nine months in 1930–31 in Germany and France, later citing important interactions with R. Courant, H. Weyl, E. Landau, C. Carathéodory, M. Frechet, P. Levy. Two years later he was appointed director of the Mathematical Research Institute at the university, a position he held until 1939 and again from 1951 to 1953. In 1938–1958 he was a head of the new Department of Probability and Statistics at the Steklov Mathematical Institute. Between 1946 and 1949 he was also the head of the Turbulence Laboratory in the Institute of Theoretical Geophysics.

Andrei Nikolaevich Kolmogorov was a Renaissance man: his first scientific work was on medieval Russian history; he then did research on metallurgy, ballistics, biology and statistics of rhythm violations in classical poetry, worked on educational reform, was the scientific head of a round-the-world oceanological expedition and used to make 40 km cross-country ski runs wearing only shorts. But first and foremost he was one of the greatest and most universal mathematicians of the twentieth century, if not of all time (Kendall et al., 1990). Kolmogorov put the notion of probability on a firm axiomatic foundation (Kolmogorov, 1933) and deeply influenced many branches of modern mathematics, especially the theory of functions, the theory of dynamical systems, information theory, logics and number theory. Seventy-one people obtained degrees under his supervision, among them several great and quite a few outstanding scientists. There is a certain grand design in the life work of Kolmogorov, to which one cannot give justice in this short essay. In his own words:

> I wish to stress the legitimacy and dignity of a mathematician, who understands the place and the role of his science in the development of natural sciences and technology, yet quietly continues to develop ‘pure mathematics’ according to its internal logics.

Kolmogorov used to claim that the mathematical abilities of a person are in inverse proportion to general human development: “Supreme mathematical genius has his development stopped at the age of five or six when kids like to tear off insect legs and wings”. Kolmogorov estimated that he himself stopped at the age 13–14 when adult problems do not yet interfere with a boy’s curiosity about everything in the world (Shiryaev, 2006, pp. 43, 171). Recall that Kolmogorov turned 14 in 1917 when the Revolution struck.

What brought this man to turbulence? Kolmogorov’s interest in experimental aspects may have been triggered as early as 1930 when he met Prandtl, as did Friedman eight years earlier (Frisch, 1995). An impetus could have been the creation of the Institute of Theoretical Geophysics by academician O. Schmidt, who in 1939 made the newly elected academician Kolmogorov a secretary of the section of Physics and Mathematics (Ya.G. Sinai, private communications, 2009–2010). Kolmogorov’s works on stochastic processes and random functions immediately predate his work on turbulence. Turbulence presents a natural step from stochastic processes (as functions of a single variable) to stochastic fields (as functions of several variables). His diary entry that starts this section may shed additional light; see also Yaglom (1994).

Kolmogorov later remarked that “it was important to find talented collaborators . . . who could combine theoretical studies with the analysis of experimental results. In this respect I was quite successful” (Yaglom, 1994). The first student of Kolmogorov to work on turbulence was the mechanical engineer M. Millionschikov who treated turbulence decay. In 1939, Loitsyansky used the Kármán–Howarth equation to infer the conservation of the squared angular momentum of turbulence: $\Lambda = \int r_{12}^2 \langle (\mathbf{v}_1 \cdot \mathbf{v}_2) \rangle d\mathbf{r}_{12}$. Considering the late (viscous) stage of turbulence decay when the size of the turbulence region grows as $l(t) \simeq \sqrt{\nu t}$, one can readily infer the law of the energy decay: $v^2(t) \simeq \Lambda l^{-5} \propto t^{-5/2}$. Also, neglecting the third moment (as had Friedman and Keller before), Millionschikov obtained a closed equation and solved it for the precise r, t dependencies of the second moment (Millionschikov, 1939). To describe turbulence at large Reynolds number Re, one needs to face eventually the third moment and account for the nonlinearity of hydrodynamics. Kolmogorov did that himself, estimating $dv^2/dt \simeq v^3/l$ and obtaining $l(t) \propto \Lambda^{1/7} t^{2/7}$ (Kolmogorov, 1941b). While Kolmogorov used his theory of small-scale turbulence (to be described below) to argue for these estimates, the relation $v \simeq l/t$ for integral quantities seems to be not very sensitive to the details of microscopic theories. The correction, unexpectedly, came from another direction: conservation of the Loitsyansky integral takes place not universally but depends on the type of large-scale correlations in the initial turbulent flow. In terms of

Fourier harmonics

$$\mathbf{v}(\mathbf{p}) = \int \mathbf{v}(\mathbf{r}) \exp(i\mathbf{p} \cdot \mathbf{r})\, d\mathbf{r},$$

the energy spectral density $E(p) = p^2|\mathbf{v}(\mathbf{p})|^2$ is expected to go to zero at $p\,l \to 0$ as an even power of p. Only if the quadratic term is absent and $E(p) \propto \Lambda p^4$ is Λ then conserved. If, however, $E(p) \propto S\,p^2$, then it is the squared momentum (called the *Saffman invariant*), $S = \int \langle(\mathbf{v}_1 \cdot \mathbf{v}_2)\rangle d\mathbf{r}_{12}$, which is conserved and determines turbulence decay. This is treated in more detail in the chapters about Batchelor and Saffman. More interesting was the second paper of Millionschikov (1941), where the quasi-normal approximation was presented (apparently formulated by Kolmogorov who often attributed his results to students; Yaglom, 1994). This approximation consists in supplementing the equation for the second moment (which contains third moment) by the equation for the third moment (which contains the fourth moment, which is decoupled via the second moments, assuming Gaussianity) (Millionschikov, 1941). Such an approximation is valid only for weakly nonlinear systems such as the weak wave turbulence described in Section 6.4 below; for hydrodynamic turbulence it is a semi-empirical approximation which was extensively used for the next forty years.

In 1941 Kolmogorov was more occupied by the behavior of $E(p)$ for $p\,l \gg 1$ and by finding the third moment exactly. Already by the end of 1939, he had outlined the scheme of the mathematical description (of what we now call the Richardson cascade) based on self-similarity and predicted that $E(p)$ for $p\,l \gg 1$ will be a power law but did not obtain the exponent (see Yaglom (1994) and Obukhov (1988, p. 83)). Some time in 1940 (most likely in the Fall), Andrei Nikolaevich invited another student, mathematician Alexander Obukhov, and suggested thinking about the energy distribution in developed turbulence. At that time, Kolmogorov did not know about the Richardson cascade picture, while Obukhov did (Golitsyn, 2009). Obukhov later recalled that they met two weeks later, compared notes and found that the exponent was the same – the first Kolmogorov–Obukhov theory (KO41[1]) came into being (Golitsyn, 2009).

Alexander Mikhailovich Obukhov was born in 1918 into a middle-class family in Saratov. He finished school in 1934 and spent a year working on a weather observation station, which probably influenced his long-life fascination with atmospheric phenomena. There, he published his first scientific work *Atmospheric turbidity during the summer drought of 1934*. The following year

[1] I use the abbreviations KO41, KO62 deliberately (rather than the more usual K41, K62) in order to highlight the importance of Obukhov's contributions, which ran in parallel with those of his mentor Kolmogorov.

he was old enough to be accepted to Saratov University where he wrote in 1937 his first mathematical work *Theory of correlation of random vectors* which received first prize in the all-country student competition (on the occasion of the jubilee of the Revolution) and attracted Kolmogorov's attention. That was an extraordinary work on multivariate statistics where the young student proposed a new statistical technique which later became known as canonical correlation analysis (simultaneously proposed by the American statistician H. Hoteling). Kolmogorov invited Obukhov to transfer to the mathematical department of Moscow University in 1939. Obukhov graduated in 1940 and was allowed to stay in the University for research work, in particular, on spectral properties of sound scattered by a turbulent atmosphere. It was then natural that Obukhov took a spectral approach to turbulence.

After that fateful meeting, when Kolmogorov and Obukhov compared notes and found that their results agreed, they published separately. The first Kolmogorov paper was submitted on 28 December 1940 (Kolmogorov, 1941a). Kolmogorov considers velocities at two points, following Friedman and Keller, whose work he knew and valued (Yaglom, 1994), yet he was apparently the first to focus on the velocity differences $\mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2$. Kolmogorov describes a multi-step energy cascade (without citing Richardson) as a "chaotic mechanism of momentum transfer" to pulsations of smaller scales. He then argues that the statistics of velocity differences for small distances (and small time differences) is determined by small-scale pulsations which must be homogeneous and isotropic (far from boundaries). That is, Kolmogorov introduces local homogeneity and turns Taylor's global isotropy (see Sreenivasan's chapter on Taylor) into local isotropy. Kolmogorov never invokes an accelerating nature of the cascade. He then makes a very strong assumption (later found to be incorrect) that the statistics of $\mathbf{v}_{12}$ at distances r_{12} much less than the excitation scale L is completely determined by the mean energy dissipation rate, defined as

$$\bar{\epsilon} = \left\langle \frac{\partial v^2}{2\partial t} \right\rangle = \frac{\nu}{2} \sum_{ij} \left\langle \left(\frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} \right)^2 \right\rangle .$$

That allowed him to define the viscous (now called Kolmogorov) scale as $\eta = (\nu^3/\bar{\epsilon})^{1/4}$ and make the second (correct) assumption that for $r_{12} \gg \eta$ the statistics of velocity differences is independent of the kinematic viscosity ν. For $\eta \ll r_{12} \ll L$, one uses both assumptions and immediately finds from dimensional reasoning that $\langle v_{12}^2 \rangle = C(\bar{\epsilon} r_{12})^{2/3}$, where the dimensionless C is called the Kolmogorov constant (even though it is not, strictly speaking, a constant, as will be clear later).

What made a mathematician hypothesize so boldly?

> I soon understood that there was little hope of developing a pure, closed theory, and because of absence of such a theory the investigation must be based on hypotheses obtained on processing experimental data. While I didn't do experiments, I spent much energy on numerical and graphical representation of the experimental data obtained by others (Kolmogorov, 2001).

Sinai recalls Kolmogorov describing how he inferred the scaling laws after "half a year analyzing experimental data" on his knees on the apartment floor covered by papers (Shiryaev, 2006, p. 207). Some thirty years later, we find Andrei Nikolaevich again in this position on the ship's cabin floor catching mistakes in the oceanic data during a round-the-world expedition (Shiryaev, 2006, p. 54). In 1941, the data apparently were from the wind tunnel (Dryden et al., 1937); they were used in the third 1941 paper (Kolmogorov, 1941c) to estimate C.

More importantly, in this third paper, Kolmogorov uses the Kármán–Howarth equation, implicitly *assumes* that, although proportional to ν, the dissipation rate $\bar{\epsilon}$ has a finite limit at $\nu \to 0$, and derives the elusive third moment. Schematically, one takes the equation of motion (6.1) at some point 1, multiplies it by $\mathbf{v}_2$ and subtracts the result of the same procedure taken at point 2. All three forces acting on the fluid give no contribution in the interval $\eta \ll r_{12} \ll L$: viscous friction because $r_{12} \gg \eta$, external force because $r_{12} \ll L$ and the pressure term because of local isotropy. This is why that interval is called inertial, the term so suggestive as to be almost misleading, as we will see later. In this interval the cubic (inertial) term, which is the energy flux through the scale r_{12}, is equal to the time derivative term, which is a constant rate of energy dissipation: $\langle(\mathbf{v}_{12} \cdot \nabla)v_{12}^2\rangle = -2\langle \partial v^2/\partial t\rangle = -4\bar{\epsilon}$. Integrating this one gets

$$\langle(\mathbf{v}_{12} \cdot \mathbf{r}_{12}/r_{12})^3\rangle = -4\bar{\epsilon}r_{12}/5 \,. \tag{6.2}$$

For many years, the so-called 4/5-law (6.2) was the only exact result in the theory of incompressible turbulence. It is the first derivation of an 'anomaly' in physics in a sense that the effect of breaking the symmetry (time-reversibility) remains finite while the symmetry-breaking factor (viscosity) goes to zero; the next example, the axial anomaly in quantum electrodynamics, was derived by Schwinger ten years later (Schwinger, 1951).

Obukhov's approach is based on the equation for the energy spectral density written as $\partial E/\partial t + D = T$ where D is the viscous dissipation and T is the Fourier image of the nonlinear (inertial) term that describes the energy transfer over scales (Obukhov, 1941). Obukhov starts his paper by saying that for a given observation scale $l = 1/p$, larger-scale velocity fluctuations provide almost uniform transport while smaller-scale eddies provide diffusion. It is then natural to divide the velocity into two orthogonal components, containing respectively large-scale and small-scale harmonics: $\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}'$. Obukhov stresses that this is not an absolute Reynolds separation into the mean and fluctuations but decomposition conditional on a scale, as a harbinger of the renormalization-group approach which appeared later in high-energy physics and critical phenomena. In this very spirit, Obukhov averages his energy equation over small-scale fluctuations, shows that what contributes to T is the product $v'v'\nabla\bar{v}$ and then decouples it as $E(p)$ times the root-mean-square large-scale gradient $\Delta^{1/2}(p)$ defined by $\Delta(p) = \int_p^\infty k^2 E(k) d^3k$. That way of closure differs from that of Friedman and Keller since the focus is not on instantaneous values and solving the initial value problem for dynamic meteorology but on average values and finding a steady-state distribution. Obukhov then solves the resulting nonlinear (but closed!) equation and finds the spectrum $E(p) \propto p^{-5/3}$ which gives a Fourier transform $\langle v_{12}^2 \rangle = (Ar_{12})^{2/3}$. For the boundary of this spectrum he finds the Kolmogorov scale (which thus should be called Kolmogorov–Obukhov scale). He then derives the law of turbulent diffusion in such a velocity, $R^2(t) = At^3$, compares it with the Richardson diffusion law $R^2(t) = \bar{\epsilon}t^3$ and obtains $A \simeq \bar{\epsilon}$. Along the way, Obukhov gives a theoretical justification for the scaling of turbulent diffusivity $D(l) \propto l^{4/3}$ that was empirically established by Richardson. Obukhov ends by estimating the rate of atmospheric energy dissipation, assuming that 2 percent of the solar energy is transformed into winds[2], and obtains a factor comparable with that measured by Richardson. Magnificent work!

One can imagine the elation the authors felt upon discovering such beautiful simplicity in such a complicated phenomenon: the universality hypothesis was supported by the exact derivation of the third moment (6.2) and by the experimental data. One is tempted to conclude that the statistics of the velocity differences in the inertial interval is determined solely by the mean energy dissipation rate. What could possibly go wrong?

The answer came from a physicist. Lev Davidovich Landau was perhaps as great and universal a physicist as Kolmogorov was a mathematician. The fundamental contributions of Landau and his school and the monumental, unique Landau–Lifshits course of theoretical physics shaped, to a significant extent, the physics of the second half of the twentieth century. Landau was born in 1908 and grew up enthusiastic about the communist ideas. The years 1929–1931 he spent abroad, interacting with N. Bohr, W. Pauli, W. Heisenberg,

[2] Obukhov does not justify the estimate: my guess is that he took 2 percent as an estimate for the relative change of the Kelvin temperature between day and night; see Peixoto and Oort (1984) for the modern data.

R. Peierls and E. Teller among others. In the mid-1930s, Landau discovered that he could no longer travel abroad. Building his school and creating the course may be seen as an attempt to create a civilization in what he saw as a wilderness. In 1938, Landau co-authored an anti-Stalin leaflet, was arrested and spent a year in Stalin's jails; after Kapitza and Bohr wrote to Stalin, Landau was freed with his black hair turned gray.

Meanwhile the Second World War eventually came to the Soviet Union and a large part of the Academy was evacuated to Kazan. There, Kolmogorov gave a talk on their results on 26 January 1942. Landau was present. An official record of the talk contains a brief abstract by Kolmogorov and a short remark by Landau. In the abstract, Kolmogorov lucidly presents his results on the local structure and then adds something new: a closed system of three partial differential equations that describe large-scale flow and integrated properties of turbulence (the energy and the strain rate). That semi-empirical model is a significant step forward compared with the earlier models of Prandtl, von Kármán and Taylor, where the Reynolds equations for mean velocity were closed by hypothetical algebraic equations for Reynolds stresses. In 1945, Prandtl suggested a less sophisticated two-equation model. The models of the type suggested by Kolmogorov (later invented independently by Saffman and others), found, with the advent of computers, numerous engineering applications. Landau remarked:

> Kolmogorov was the first to provide a correct understanding of the local structure of turbulent flow. As to the equations of turbulent motion, it should be constantly born in mind . . . that in a turbulent stream the vorticity is confined within a limited region; qualitatively correct equations should lead to just such a distribution of eddies.

It is reasonable to assume that the second part of Landau's remark is related to the second part of Kolmogorov's presentation, i.e. to the equations for the large-scale flows. In 1943 Landau derived his exact solution for a laminar jet from a point source inside a fluid (Landau and Lifshitz, 1987), so apparently he was thinking about flows of different shapes. Incidentally, I was unable to find a steady solution of Kolmogorov's equations that describe such a limited region. One may try to interpret Landau's remark as implicitly questioning the universality of small-scale motions: the further the probe from the axis of a turbulent jet, the less time it spends inside the turbulence region because of boundary fluctuations; therefore, the value of the Kolmogorov constant C must depend on the distance to the axis (Frisch, 1995). However, Kolmogorov explicitly postulated that his theory works away from any boundaries, so that the universal value of C is what he expects to be measured near the jet axis or deep inside other turbulent flows. It is likely that Landau started to have doubts about Kolmogorov's description of small-scale structure only later. In 1944, the sixth volume of the Landau–Lifshitz course, *Mechanics of Continuous Media*, appeared (Landau and Lifshitz, 1987). This book firmly set hydrodynamics as part of physics. The book contained a remark (attributed in later editions to Landau, 1944), which instantly killed the universality hypothesis:

> It might be thought that the possibility exists in principle of obtaining a universal formula, applicable to any turbulent flow, which should give $\langle v_{12}^2 \rangle$ for all distances r_{12} small compared with L. In fact, however, there can be no such formula, as follows from the following argument. The instantaneous value of v_{12}^2 might in principle be expressed in a universal way via the energy dissipation ϵ at that very moment. However, averaging these expressions is dependent on the variation of ϵ over times of large-scale motions (scale L), and this variation is different for different specific flows. Therefore, the result of the averaging cannot be universal.

Let us observe a moment of silence for this beautiful hypothesis.

To put it a bit differently: the third moment (6.2) is linearly proportional to the dissipation rate ϵ and is then related in a universal way to the mean dissipation rate $\bar{\epsilon}$. Yet other moments $\langle v_{12}^n \rangle$ are averages of nonlinear functions of the instantaneous value ϵ, so that their expressions via the mean value $\bar{\epsilon}$ depend on the statistics of the input rate determined by the motions at the scale L (that was more clearly formulated later by Kraichnan; see the chapter by Eyink and Frisch). The question now is whether such an influence of large scales changes factors that are of the order of unity (say making C non-universal) or changes the whole scale dependence of the moments, since now one cannot rule out the appearance of the factor (L/r_{12}) raised to some power. Kolmogorov and Obukhov themselves found the answers twenty years later, as will be described below.

During the war years the works of Kolmogorov and Obukhov were unknown to the rest of the world, becoming known after the war primarily through their discovery by Batchelor. Kolmogorov's 4/5-law was not independently derived but his 1/3-law and Obukhov's 5/3-law were rederived by Heisenberg, Weizsäcker and Onsager (Battimelli and Vulpiani, 1982; Frisch, 1995; Sreenivasan and Eyink, 2006). Apparently it is more difficult to get the factor than the scaling, all the more so because the factor is exact while the scaling is not. Note, however, that it is reasonable to expect that the moments depend in some regular way on the order n of the moment. If so, then the fact that $n = 2$ is not far from $n = 3$ means that KO41, which is exact for $n = 3$, must work reasonably well for $n = 2$, i.e. for the energy spectrum, which is indeed what measurements show. This is the reason that this flawed theory turned out to be very useful in numerous geophysical and astrophysical applications as long as one is interested in the energy spectrum and not high moments or strong fluctuations. For the next twenty years, Kolmogorov and Obukhov developed the applications and generalizations of KO41 instead of looking for a better theory. In retrospect, that seems to be a right decision. Its implementation involved the creation of a scientific school.

6.3 Applied mathematicians

One of my students rules the Earth atmosphere, another – the oceans.
A.N. Kolmogorov

Alexander Obukhov was soon joined by Andrei Monin and Akiva Yaglom, the two other key people that established the Kolmogorov school of turbulence. Andrei and Akiva were born the same year, 1921, and died the same year, 2007. They wrote the book (Monin and Yaglom, 1979) that for several decades was "the Bible of turbulence". The triple A of Alexander, Andrei and Akiva represented very different, and in some respects polar opposite, people. Alexander and Akiva were never Party members, with the latter even refusing to work on the nuclear project since he disliked the idea of developing a bomb for Stalin (Shiryaev, 2006, p. 440), while Andrei was a devoted communist who joined the Party during the war. That was a stark difference in the Soviet Union back then. Kolmogorov himself was not a Party member, yet allowed neither regime critique nor political conversations in his presence (Shiryaev, 2006, p. 442); descent from nobility and homosexuality (criminal under Soviet penal code) added extra vulnerability for Andrei Nikolaevich in Stalin's Russia.

Yaglom grew up in Moscow where his high-school friend was Andrei Sakharov (who later became a friend of Obukhov too). Akiva had a twin brother Isaak, with whom he shared a first prize at the Moscow Mathematical Olympiad in 1938. The prize was presented by Kolmogorov who never forgot good students (Yaglom, 1994) and in 1943 invited Yaglom to work on the theory of Brownian motion. Andrei Monin graduated in 1942 and the same year was also invited by Kolmogorov to work on probability distributions in functional spaces (where there is no volume element and thus no density). Both Akiva (in 1941) and Andrei (in 1942) volunteered for military service to fight in the war. Akiva was rejected because of poor eyesight. Andrei was drafted and spent the war as an officer-meteorologist serving at military airfields. He returned in 1946 ready to work on turbulence.

The first new result after 1941 was, however, obtained by Obukhov whose Kazan years were important and formative. In addition to Landau, he interacted there with the physicist M.A. Leontovich, a man of great integrity (who, among many other things, published with Kolmogorov the paper on Brownian motion in 1933). Landau and Obukhov were the first to suggest independently the Lagrangian analog of KO41. If $\mathbf{R}(t)$ describes the trajectory of a fluid particle, then the Lagrangian velocity is defined as $\mathbf{V}(t) = \mathbf{v}(\mathbf{R}, t)$. The relation $\mathbf{V}(t) - \mathbf{V}(0) \simeq (\epsilon t)^{1/2}$ first appeared in the Landau–Lifshitz textbook in 1944. Note however that the exact Lagrangian relation, which is a direct analog of the flux law (6.2), is not the (still hypothetical) two-time single-particle relation $|\mathbf{V}(t) - \mathbf{V}(0)|^2 \simeq \epsilon t$, but the Lagrangian time derivative of the *two-particle* velocity difference: $\langle d|\delta\mathbf{V}|^2/dt\rangle = -2\bar{\epsilon}$ (note that $\epsilon > 0$ in 3D and $\epsilon < 0$ in 2D) (Falkovich et al., 2001).

From 1946, Kolmogorov arranged a bi-weekly seminar on turbulence which was a springboard for the explosive development of KO41 and its applications. Obukhov started to work on the atmospheric boundary layer and dynamic meteorology. Already in 1943 he wrote a paper which because of the war was published in 1946 and yet was ahead of its time (Obukhov, 1988, p. 96; translated in Obukhov, 1971). Following Prandtl and Richardson, Obukhov considered the influence of stable stratification on turbulence. It is clear that turbulence disturbs stable stratification and increases the potential energy, thus decreasing the kinetic energy of the fluid. In other words, stratification suppresses turbulence. On the other hand, turbulence influences the vertical profile of the temperature. Obukhov developed a semi-empirical approach based on a systematic use of universal dimensionless functions. In addition to the dimensionless Richardson number that quantifies the relative role of stratification and wind shear, Obukhov measured the height in units of the sub-layer where the Richardson number is small and stratification is irrelevant. This defines what is now called the Obukhov–Monin scale, since the idea of the sub-layer was systematically exploited by Obukhov and Monin in 1954. In the paper which is a sequel to that of Obukhov (1988, p. 135), they showed that the profiles of the wind and the temperature are determined by the vertical fluxes of the momentum and heat; see Yaglom (1988) for more details.

The year 1949 was exceptionally productive. Kolmogorov applied KO41 to the problem of deformation and break-up of droplets of one liquid in a turbulent flow of another fluid: flow can break the droplet of the size a if the pressure difference due to flow $\rho(\delta v)^2 \simeq \rho(\bar{\epsilon}a)^{2/3}$ exceeds the surface tension stress σ/a (Kolmogorov, 1949). Obukhov established the basis of dynamic meteorology by his famous work on a geostrophic wind, derived what is now called the Charney–Obukhov equation for the rotating shallow water, known as Hasegawa–Mima for magnetized plasma (though I've heard Obukhov remarking that the plasma version was known to Leontovich before). Turbulence theory was significantly advanced when Obukhov published a pioneering work on the statistics of a passive scalar θ mixed by a turbulence flow (Obukhov, 1949a). Obukhov correctly describes the common action of turbulent mixing and molecular diffusion as a mechanism of relaxation. He then focuses on θ^2 (assuming $\langle\theta\rangle = 0$) which is a nontrivial step, missed by several people who got wrong answers; see Monin and Yaglom (1979). Obukhov identifies θ^2 as an analog of the energy density, arguing that when θ is the temperature then $\int \theta^2(\mathbf{r})\,d\mathbf{r}$ is the maximal work one can extract from an inhomogenously heated body. That opens the way to considering the cascade of this quantity in a direct analogy with the energy cascade. Obukhov's work then follows Kolmogorov's approach of his first 1941 paper, that is it considers the statistics of the differences $\theta_{12} = \theta_1 - \theta_2$. Obukhov assumes that there exists an interval of scales between the scales of production and dissipation where the statistics of θ_{12} is completely determined by the dissipation rates ϵ and $N = \langle \partial\theta^2/\partial t\rangle$. Dimensional reasoning then gives $\langle\theta_{12}^2\rangle \simeq N(r_{12}^2/\epsilon)^{1/3}$. Of course, the right-hand side here is the mean dissipation rate of θ^2 multiplied by the typical turnover time on the scale r_{12}. This 2/3-law was independently established by Corrsin in 1951 and is called the Obukhov–Corrsin law (see also the Corrsin chapter). The second exact relation in turbulence theory, the flux expression for a passive scalar analogous to (6.2) for energy, was derived by Yaglom the same year (Yaglom, 1949a). That same year Obukhov dispelled an erroneous belief (expressed in Millionschikov, 1941) that pressure fluctuations are zero in incompressible turbulence (Obukhov, 1949b). By taking the divergence of the Navier–Stokes equation, Obukhov obtained the incompressibility condition $\Delta p = -\nabla_i\nabla_j(v^i v^j)$, which allows one to express the second moment of pressure via the fourth moment of velocity, which is then decoupled via the product of the second moments, again assuming Gaussianity: $\langle p_{12}^2\rangle \propto r^{4/3}$. That 4/3-law together with 5/3, 2/3 and others was the basis for the joke that Obukhov discovered the fundamental 'all-thirds law'. There is a truth in every joke since the number 3 in the denominator of these scaling exponents arises because of two fundamental reasons: (i) the nonlinearity of the equation of motion is quadratic and (ii) the fluxes considered are of the quadratic integrals of motion. Immediately, Yaglom used Obukhov's approach to derive the mean pressure gradient and the mean squared fluid acceleration (Yaglom, 1949b). Remarkably, Yaglom's estimate for atmosphere showed that typical winds can make for accelerations exceeding that of gravity. Obukhov, Monin and Yaglom had a chance to experience that, flying on balloons in turns, thus continuing Friedman's tradition; in 1951 the wind data were obtained confirming KO41 scaling (Obukhov, 1951) (later, they also observed a layered structure of turbulence, the so-called turbulent 'pancakes', predicted by Kolmogorov in 1946 – Shiryaev, 2006, p. 181). In 1951, Obukhov and Yaglom published together a detailed paper that presented all the results on pressure and acceleration. Similar results were obtained independently by Heisenberg in 1948 and Batchelor in 1951.

The Kolmogorov turbulence seminar was attended by applied scientists and engineers as well, and discussions of applied problems went along with the focus on fundamental issues. In 1951, Kolmogorov accepted the next student, Gregory Barenblatt, whose name he remembered from the list of the students whose work won first prizes (following the familiar Obukhov–Yaglom pattern). Barenblatt was given the task of describing the transport of a suspended sediment by turbulent flows in rivers. Somewhat similarly to stably stratified flows, turbulence spends energy lifting sediments which, being small, then dissipate energy into heat when descending. Barenblatt built an elegant theory similar to that of Obukhov–Monin (Barenblatt, 1953).

Important insights into the advection mechanisms were obtained by eliminating global sweeping effects and describing the advected fields in a frame whose origin moves with the fluid. This picture of the hydrodynamic evolution, known under the name of quasi-Lagrangian description, was first introduced in Monin (1959). In a kind of a bridge between work on stratification and passive scalars, Obukhov considered unstable stratification, accounted for the buoyancy force and defined a new scale above which this force starts to be important (Obukhov, 1959). Bolgiano discovered this independently the same year and also suggested KO41-type scaling for turbulent convection at larger scales (Bolgiano, 1959).

In 1956 the Institute of Geophysics was divided into three parts and Obukhov was appointed director of the newly created Institute of Atmospheric Physics which now bears his name. That followed his long conversation with Leontovich which ended with the advice to "avoid administrative zeal" (Obukhov, 1990). In the Soviet Union, the Academy was a huge body that operated hundreds of scientific institutes with tens of thousands of researchers. Academic institutes worked under the strict Party control and a non-communist director was a rare bird. Obukhov flouted Party policy in another important respect: employing numerous Jewish scientists in his Institute. Since the late 1940s anti-Semitism as a Party policy was steadily gaining ground in Russian society and academia. Moscow University was particularly hostile: it was difficult for a Jew to be accepted as an undergraduate and next to impossible as a graduate student; this situation further deteriorated at the end of 1960s when undergraduate studies were closed as well (all the way to the 1970s when I avoided Moscow and went to Novosibirsk University). The mathematical students of Kolmogorov were particularly affected. For example, Sinai was not accepted for graduate studies after the committee failed him in Marxist philosophy: Kolmogorov was present at the exam but did not interfere (Ya.G. Sinai, private communications, 2009–2010). Kolmogorov then negotiated for Sinai a second attempt which succeeded. Turbulence researchers had it easier thanks to the Institute of Geophysics and later to Obukhov's Institute. Remarkably, that quite unusual director did not even fire refuseniks as was required by a direct Party order. Obukhov was universally admired by his co-workers despite his sometimes harsh style (its acceptance was softened by a common agreement that he was invariably the smartest person in the room, best equipped to "rule the Earth's atmosphere").

Andrei Monin was appointed "to rule the oceans" in 1965 when he was made director of the Institute of Oceanology. He had not only been a devout Party member since 1945, but a high-level if somewhat reluctant (Golitsyn, 2009) functionary in the Party hierarchy as an instructor and then the deputy chairman of the Science Department of the Party Central Committee. While the Academy kept some marginal degree of independence in electing (or rejecting) new members, the Department was the body which actually set the policy, appointed directors, issued permits for visits abroad etc. During the 1950s, the Department was particularly hostile towards "the group of non-communist scientists led by Tamm, Leontovich and Landau" (Monin, 1958).

Around 1960–61, Obukhov decided at last to address Landau's remark on dissipation rate fluctuations and initiated theoretical and experimental investigations into the subject (Golitsyn, 2009). Systematic measurements of wind velocity fluctuations were made by Gurvitz (1960). The calculations of the fluctuations of the energy dissipation rate ϵ, assuming quasi-normality, was done by Obukhov's student G.S. Golitsyn, who later extended the approach of KO41 to the analysis of the dynamics of planetary atmospheres (Golitsyn, 1973) and succeeded Obukhov as director of the Institute. Experimental data had shown that fluctuations were much stronger than the theoretical estimates. Strong non-Gaussianity of velocity derivatives was also observed before by Batchelor and Townsend. Looking for an appropriate model for the statistics of ϵ, Obukhov turned to another seminal paper of 1941 by Kolmogorov (1941d) on a seemingly different subject: ore pulverization. Breaking stones into smaller and smaller pieces presents a cascade of matter from large to small scales. A stone that appears after m steps is of size ϵ_m, which is a product of the size ϵ of an initial large stone and m random factors of fragmentation: $\epsilon_m = \epsilon e_1 \dots e_m$, where $e_i < 1$. If those factors are assumed to be independent, then $\log \epsilon_m$ is a sum of independent random numbers. As m increases, the statistics of the sum tends to a normal distribution with the variance proportional to m. In other words, multiplicative randomness leads to log-normality.

Since the number of steps of the cascade from L to r is proportional to $\ln(L/r)$, Obukhov then assumed that the energy dissipation rate that is coarse-grained on a scale r has such a log-normal statistics with variance $\langle \ln^2(\epsilon_r/\bar{\epsilon})\rangle = B + \mu \ln(L/r)$, where B is a non-universal constant determined by the statistics at large scales. Note that the variance grows when r decreases and so also do other (not very high) moments: $\langle \epsilon_r^q \rangle \propto (L/r)^{\mu q(q-1)/2}$. Obukhov then formulated the refined similarity hypothesis: KO41 is true locally; that is, the velocity difference at a distance r is determined by the dissipation rate coarse-grained on that scale: $\delta v(r) \simeq (\epsilon_r r)^{1/3}$. Averaging this expression over the log-normal statistics of ϵ_r one obtains new expressions for the structure functions that contains non-universal factors C_n and universal exponents: $\langle v_{12}^n \rangle = C_n r^{n/3} (L/r)^{\mu n(n-3)/18}$. This general formula was actually derived by Kolmogorov who was shown the draft of Obukhov (1962) (containing only $n = 2$) before boarding the train that took him to the Marseille conference, separately from rest who flew there. Kolmogorov arrived at Marseille with his own draft (Kolmogorov, 1962) and their two presentations were highlights of the conference. The Marseille gathering between 28 August and 3 September 1961 was a remarkable event that brought almost all the leading researchers together, many for the first time. Yaglom recalls:

> The USSR delegation included Kolmogorov . . . , his two pre-war students M.D. Millionschikov and A.M. Obukhov, and me – a war-years student. Such a composition had the flavor of Khrushchev's liberalization (for me it was the first time I was permitted to attend a meeting in a 'capitalist country').

Russians at last had a chance to meet turbulence's great scholars from all generations. Most of the heroes of this book were present: von Kármán, Taylor, Batchelor, Townsend, Corrsin, Saffman and Kraichnan. It is poignant to see Kolmogorov and Kraichnan (whose names are forever linked by the 2D–3D 5/3-scaling) in the same photograph.

The new theory KO62 gives the same linear scaling for the third moment. Attempts to estimate μ from experimental data on the variance of dissipation or velocity structure functions give $\mu \simeq 0.2$, so that KO62 only slightly deviates from KO41 for $n < 10 \div 12$. Its importance must be then mostly conceptual. The main point is understanding that the relative fluctuations of the dissipation rate grow unboundedly with the growth of the cascade extent, L/r (in his paper, Kolmogorov credits that to Landau even though the latter's 1944 remark did not mention any scale-dependence of the fluctuations – Frisch, 1995). That understanding opened the way to the description of dissipation concentrated on a measure (Novikov and Stewart, 1964), which was later suggested to be fractal (Mandelbrot, 1974), and shown to be actually multi-fractal (Parisi and Frisch, 1985; Meneveau and Sreenivasan, 1987). Let us stress another

Figure 6.1 Kolmogorov and others at the Marseille conference.

conceptual point: the 5/3-law for the energy spectrum is incorrect despite being (outside of the turbulence community) the most widely known statement on turbulence. Still, KO62 does not seem to be such a momentous achievement as KO41. First, it evidently does not make sense for sufficiently high n. Second and more important, it is still under the spell of two magic concepts of the Kolmogorov school: Gaussianity and self-similarity. Compared with KO41, the new version KO62 somehow pushes these two further down the road: the new (refined) self-similarity is local and Gaussianity is transfered to logarithms, replacing additivity with multiplicativity. Still, KO62 is based on the belief that a single conservation law (of energy) explains the physics of turbulence and that the (local) energy transfer rate completely determines local statistics. As we now believe, direct turbulence cascades (from large to small scales) have, at a fundamental level, nothing to do with either Gaussianity or self-similarity, even though these concepts can help to design useful semi-empirical models for applications. There is more to turbulence than just cascades. Energy conservation determines only a single moment (the third for incompressible turbulence). To understand the nature of turbulence statistics, one returns to the old remark of Friedman that the correlation functions are "moments of conservation". In this way, one discovers an infinite number of statistical conservation laws having a geometrical nature, each determining its own correlation function; to this must be added that the exponents are now measured with higher precision and they are neither KO41, nor KO62; see for example Falkovich et al. (2001) and Falkovich and Sreenivasan (2006).

Note in passing that the interaction between Landau and Kolmogorov was a two-way street. We described above how Landau's reaction to KO41 changed the theory of turbulence. No less fruitful was Kolmogorov's reaction to Landau's suggestion in 1943 that, as the Reynolds number Re grows, the sequence of instabilities leads to multi-periodic motion; that is, the attractor in the phase space of the Navier–Stokes equation is a torus whose dimensionality grows with Re. Superficially, this seems to be very much in the spirit of Kolmogorov's own 1941 argument that "at large Re, pulsations of the first order are unstable in their own turn so that the second-order pulsations appear" (Kolmogorov, 1941a). However, Kolmogorov developed deeper insights into the onset of turbulence and posed the question of whether it is possible that a continuous spectrum appears at finite Re. That was answered by work on dynamical systems theory, which he started in 1953 "because the hope appeared and my spirit uplifted" (Stalin died). The resulting KAM theory (after Kolmogorov, Arnold and Moser) describes which invariant tori survive under a slight change of Hamiltonian and forms the basis of understanding Hamiltonian chaos. Later, Kolmogorov initiated a great synthesis of the random and deterministic, based on the notions of entropy and complexity, magnificently carried out by his student Sinai and others. To overcome the natural prejudice of considering dynamic systems as deterministic, one needs to be profoundly aware of the finite precision of any measurement and of the exponential divergence of trajectories (Kendall et al., 1990). Kolmogorov–Sinai entropy and dynamical chaos are fundamental to our understanding of numerous phenomena; in particular, related ideas were used later for describing the statistics of turbulence below the Kolmogorov–Obukhov scale where the flow is spatially smooth but temporally random; see for example Falkovich et al. (2001). In addition, Kolmogorov's program for the 1958 seminar included the task of developing the theory of one-dimensional (Burgers) turbulence which was completed by Sinai and others some 40 years later.

I find it puzzling though that Kolmogorov himself never applied his powerful probabilistic thinking and understanding of stochastic processes and complexity to quantum mechanics and statistical physics (it was done by his students Gelfand and Sinai respectively). It seems that Kolmogorov's direct contact with physics was only via classical mechanics and hydrodynamics (Novikov, 2006).

Obukhov started a new chapter in Obukhov (1969) by introducing what he called systems of hydrodynamic type and what were later known as shell models. He was inspired by the 1966 work of Arnold on the analogy between the

Euler equation for incompressible flows and the Euler equation for solid body motion; see Arnold and Kesin (1998) for the detailed presentation[3]. Obukhov approximated fluid flow by a system of ordinary differential equations with quadratic nonlinearity and quadratic integrals of motion. Since there was no consistent way of determining the number of equations for this or that type of flow, Obukhov initiated laboratory experiments and their detailed comparison with computations. It is worth noting that Obukhov and his co-workers worked on few-mode dynamic models (apparently independently of E. Lorentz) as well as on chains intended to model turbulence cascades (Gledzer et al., 1981).

We conclude this section by referring the reader to the magnificent opus by Monin and Yaglom where much more can be found on KO41, KO62 and many other subjects including the field-theoretical approaches of Edwards, Kraichnan and others. "If ever a book on turbulence could be called definitive", declared *Science* in 1972, "it is this book by two of Russia's most eminent and productive scientists in turbulence, oceanography, and atmospheric physics." As does the presentation here, it stresses the physics of KO41 and KO62, but also makes it clear that the theory in its entirety is definitely that of mathematicians. The mathematical foundations were laid before and after 1941 in the works of Kolmogorov, Obukhov, Gelfand, Yaglom and others. A complete analysis of stationary processes using the Hilbert space formulation was done in 1941. Considerable work was done on spectral representations of random processes; subtle points of legitimacy and convergence were cleared for the Fourier transform and other orthogonal expansions for translation-invariant random functions, which physicists take for granted without much thought. Part 2 of the Monin–Yaglom book was finished in 1966 and published in 1967, in time to cite the first 1965 paper of Zakharov on wave turbulence. That is the subject of the next section.

6.4 Theoretical physicist

Keep your hands off our light entertainment,
Do not tempt us with crumbs of attainment,
Do not teach us the right aspirations,
Do not tease us with serving the nation.
V. Zakharov (2009b)

Another stream in Russian work on turbulence originates from the Landau school. Apart from his cameo appearance in the Kolmogorov–Obukhov

[3] I believe that exploration of this analogy will bear even more fruit in the future.

part of the story, Landau himself didn't work on the theory of developed turbulence, despite his firm belief that the problem belongs in physics. In the 1950s, he was interested in plasma physics and steered in this direction the young Roald Sagdeev, who went to work in the theoretical division of the Russian project on controlled thermonuclear fusion. Plasmas are subject to various instabilities and practically always are turbulent. Inspired by the works of David Bohm on an anomalous diffusion in plasmas (Bohm et al., 1949) and the needs of thermonuclear fusion, the theory of plasma instabilities and turbulence was intensely developed in Russia by B. Kadomtsev, A. Vedenov, E. Velikhov and R. Sagdeev during the 1950s and 1960s. Sagdeev's uniform approach to plasma hydrodynamics (extended then to other continuous media) was a trademark of the Landau school: at first all dynamical equations of continuous media were supposed to be written in a canonical Hamiltonian form, then particular solutions are found and their stability analyzed, then perturbation theory applied to the description of random fields.

To carry on this project, a most unlikely figure appeared: a student expelled for a fistfight from the Moscow Energy Institute. Vladimir Zakharov was born in Kazan in 1939 to the Russian family of an engineer. When at elementary school, he did well and had a slight burr so was considered a Jew by his peers – an experience conducive to an early formation of personal independence. Zakharov knew Sagdeev first as a friend of his older brother and he met him in the Energy Institute where Sagdeev was teaching physics part-time. After expulsion, Sagdeev brought Zakharov to G. Budker who led parallel experimental projects in two fields (high energy and plasma physics) and two cities (Moscow and Novosibirsk). In 1957, a new scientific center was created some 3500 kilometers east of Moscow. In 1961 Budker convinced Sagdeev and Zakharov to leave Moscow and come to that new center in Novosibirsk. Sagdeev was to lead the plasma physics department in the newly established Nuclear Physics Institute (now the Budker Institute) while Zakharov was admitted to Novosibirsk University, leaving all his troubles behind and starting a new life in the brave new world of hastily built barrack-style buildings in the middle of the taiga.

Note in passing that Zakharov's poetry was published by the main Russian literary magazines, included in anthologies etc.: there exists a bilingual book with English translations (Zakharov, 2009a). As a scientist, he grew up inside a strongly interacting community of physicists and mathematicians, particularly influenced by M. Vishik, V. Pokrovsky and G. Budker. Zakharov succeeded in making important advances in the directions usually considered far apart: integrability and exact solutions on the one hand, and turbulence on the other. In particular, he was able to find turbulence spectra as exact solutions.

Following Sagdeev's program, Zakharov reformulated the equations for plasma and water waves in terms of Hamiltonian variables. Written as the amplitudes of plane waves, all such equations have the form $\dot{a}_k = -i\omega_k a_k +$ nonlinear terms (quadratic, cubic, etc.). Now, if the wave amplitudes are small while the frequencies ω_k are large and different for different k, one can treat nonlinear terms as small perturbations. Considering a set of random small-amplitude waves in a random-phase approximation, one expresses the time derivative of the second moment, $\langle a_k a^*_{k'} \rangle = n_k \delta(\mathbf{k} - \mathbf{k}')$, via the third moment, which is relevant if three-wave resonances are possible; i.e. one can find triads of wave vectors such that $\omega_{k+k'} = \omega_k + \omega_{k'}$. Exactly as in a quasi-normal approximation, one then writes the equation for the third moment, decouples the fourth moment and obtains the kinetic equation for waves:

$$\dot{n}_k = \int W(\mathbf{k}, \mathbf{p}, \mathbf{q}) \delta(\omega_k - \omega_p - \omega_q) \delta(\mathbf{k} - \mathbf{p} - \mathbf{q})(n_p n_q - n_k n_p - n_k n_q) d\mathbf{p} d\mathbf{q}$$
$$+ \text{ cyclic permutations } k \to p \to q \to k.$$

The right-hand side is a collision term very much like in the Boltzmann kinetic equation. The idea of phonon collisions was introduced in Peierls (1929); the collision term was used by Landau and Rumer in 1937 to calculate sound absorption in solids (Landau and Rumer, 1937). In the article (Zakharov, 1965) submitted on 28 October 1964, Zakharov took this equation (which he learnt from Camac et al., 1962) and asked if it has a stationary solution different from the equilibrium Rayleigh–Jeans distribution $n_k = T/\omega_k$. Inspired by the Kolmogorov–Obukhov spectrum he set to look for a power-law solution $n_k \propto k^{-s}$. Taking first the case of acoustic waves when the coefficients are relatively simple, $\omega_k \propto k$ and $W \propto kpq$, Zakharov first checked that the collision integral tends to $-\infty$ when $s \to 4$ and to $+\infty$ when $s \to 5$, so it has to pass through zero at some intermediate s. He then bravely substituted $s = 4.5$ and obtained for the collision integral 18 gamma-functions that promptly canceled each other. The first Kolmogorov–Zakharov spectrum was born. Still, it took some time for Zakharov to appreciate that the spectrum indeed describes a cascade of energy local in k-space and is an exact realization of KO41 ideas: by checking the convergence of the integrals in the kinetic equation at $p \to 0$ and $p \to \infty$, one can directly establish that the ends of the inertial interval really do not matter, in contrast with hypotheses about turbulence of incompressible fluids. Interestingly, the position of the Kolmogorov–Zakharov exponent exactly in the middle of the convergence interval is a general property now called counterbalanced locality: the contributions of larger and smaller scales are balanced on the steady spectrum (Zakharov et al., 1992). In 1966, Zakharov submitted his PhD thesis (under the supervision of Sagdeev) which was devoted to waves on a water surface (Zakharov, 1966; Zakharov and Filonenko, 1966). There, one finds a complete description for the case of capillary waves: obtaining the spectrum from the flux constancy condition, checking locality as integral convergence and showing that this is indeed an exact solution by using conformal transforms that were independently invented by Kraichnan for his direct interaction approximation at about the same time (see Chapter 10). Then Zakharov takes on the turbulence of gravity waves whose dispersion relation, $\omega_k \propto \sqrt{k}$, does not permit three-wave resonances. In this case, the lowest possible resonance corresponds to four-wave scattering. Every act of scattering conserves not only the energy, $E = \int \omega_k n_k d\mathbf{k}$, but also the wave action, $Q = \int n_k d\mathbf{k}$, which can be also called 'number of waves'. The situation is thus similar to the two-dimensional Euler equation which conserves both the energy and squared vorticity. In his thesis, Zakharov derives two exact steady turbulent solutions of the four-wave kinetic equation, one with the flux of E and another with the flux of Q. He then argues that the energy cascade is direct, i.e. towards small scales. While Zakharov derived an exact solution that describes an inverse cascade, he didn't explicitly interpret it as such (he also gave some arguments in the spirit of Onsager about transport of Q to large scales in a decaying turbulence). After Kraichnan's 1967 paper was published and brought to his attention by B. Kadomtsev, Zakharov realized the analogy and interpreted the spectra he derived as a double-cascade picture. In 1967, he published the direct-cascade spectrum for Langmuir plasma turbulence (Zakharov, 1967), where the inverse-cascade spectrum was obtained in 1970 by E. Kaner and V. Yakovenko from the Kharkov branch of the Landau school (Kaner and Yakovenko, 1970). Note that the hypotheses Kolmogorov formulated in 1941 are true for Zakharov's direct and inverse cascades of weak wave turbulence and are probably true for Kraichnan's inverse cascade in incompressible two-dimensional turbulence as well. In 2006, Kraichnan and Zakharov were together awarded the Dirac medal for discovering inverse cascades.

The early years of the Novosibirsk scientific center were also the years of Khrushchev's brief thaw. At that time, there was probably no other place in the country where academicians, professors and young students lived in such a close proximity and had so few barriers for scientific and social interaction. A small town in the forest, "Siberia's little Athens", was for a while allowed some extra degrees of freedom. That was about to end in 1968 when Zakharov became one of the initiators and signatories of the open letter to the Party Central Committee protesting the arrests of dissidents. But Brezhnev's time was vegetarian compared with that of Stalin: Zakharov's only punishment was a ban on foreign travel, then thought to be forever.

Hard is Athenian mien,
harder still 'midst feasting vultures.
He who will get on the wing,
sees half the world as his home.
Zakharov (2009b)

6.5 Epilogue

In twenty years no one will know what
actually happened in our country.
A.N. Kolmogorov, 1943 (Nikol'skii, 2006)

Our story ends (somewhat arbitrarily) in 1970. What followed – study of shell models by Obukhov's school, development of the weak turbulence theory by Zakharov's school, works on Lagrangian formalism and zero modes – deserves a separate essay which may be too early to write.

In his old age Kolmogorov suffered from Parkinson's disease and from an eye illness that made him almost blind. Nevertheless, he tried to work practically until the end, always surrounded by his former students, who also took turns in providing necessary help. Landau was seriously injured in an automobile accident in 1962; he was 59 days in a coma and survived with the help of his students and colleagues in the country and abroad; he lived for six more years but was unable to work. Obukhov and Yaglom worked until their last days, monuments of unaging intellect.

Kolmogorov died in 1987 and Obukhov in 1989. That year, the Berlin Wall fell, the Soviet Union opened the gates and disintegrated within two years. An exodus of scientists brought substantial parts of the Kolmogorov, Obukhov and Landau schools to the West. These schools then turned international but also weakened their links to Russia and started to lose their distinct Russian spirit.

Under one of the most oppressive regimes in the twentieth century, in the country which lost most of its educated class to emigration, civil war and terror, and which was often plagued by war, diseases, poverty and hunger, great mathematical and physical schools flourished. Scholars raised in these schools had a specific code of behavior. Long corridor chats were the most effective forums of exchanging the latest ideas. Most seminars had no sharply defined ends, some even had no clear beginning, as the people came before to discuss related subjects (Ya.G. Sinai, private communications, 2009–2010). Everyone worked inside a coherent group of people familiar with the details of each other's work (a downside was that some people never had much incentive to learn how to present their results to the outside world). Much has been said about the aggressive style and interruptions at Russian seminars. One must however understand the context: in a life which was a sea of official lies, doing science was perceived as building a small solid island of truth; even unintentional errors risked decreasing the solid ground on which we stand. Landau used to say: "An error is not a misfortune, it is a shame". One is reminded of monastic orders that preserved and advanced knowledge during the dark ages (though in other respects, most Soviet scientists weren't monks). A more prosaic reason that bonded people within a school was an impaired mobility of scientists – recall that both Kolmogorov and Landau had a postdoctoral period abroad, a possibility denied to most of their students[4]. Still, the main attraction of the schools was the personalities of the leaders.

By radically restricting creative activities, a tyrannical society channeled the creative energy into the narrow sector of natural sciences and mathematics. Russian society is more open now, and the choice of science as one's occupation is rarely placed in the context of morality. Will we ever again be blessed with universalist geniuses of the caliber of Kolmogorov and Landau?

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[4] I believe that impaired mobility was the main reason why Soviet science as a whole never lived up to our expectations.

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