Geophysical turbulence

1983 Russ. Math. Surv. 38 127

(http://iopscience.iop.org/0036-0279/38/4/R09)

Uspekhi Mat. Nauk 38:4 (1983), 113-131 Russian Math. Surveys 38:4 (1983), 127-149

Geophysical turbulence

A.S. Monin

CONTENTS

Introduction

In the school of Kolmogorov the theory of turbulence is developed like statistical hydromechanics [1], in which hydrodynamic fields describing turbulent flows of fluids or gases and satisfying the equations of hydrothermodynamics (in the case of an incompressible fluid the velocity field **u** provides a complete description of the flow, while in the case of compressible fluids or gases fields of independent thermodynamic characteristics are employed along with it) are regarded as random functions of the space-time points $M = (\mathbf{x}, t)$, and the mathematical expectation is used as the averaging operation for them (from this point of view every theory of turbulence is statistical, and the division in the literature of turbulence theories into semi-empirical theories, statistical theories, and other theories does not make sense).

§1. Statistical hydromechanics

A complete statistical description of the random functions $u_1(M), \ldots, u_k(M)$ involves specifying all the finite-dimensional probability distributions for the values of these functions $u_{j_1}(M), \ldots, u_{j_n}(M)$ on all possible finite sets of points $M_1(\mathbf{x}_1, t_1), \ldots, M_n(\mathbf{x}_n, t_n)$ in space-time. For simplicity we shall assume below that these distributions have probability densities $p_{M_1 \ldots M_n}(u_{j_1}, \ldots, u_{j_n})$ (although in the case of intermittent turbulence they can have terms in the form of delta-functions corresponding to zero values of turbulent fluctuations). The corresponding equations of evolution for them can be derived from the equations of hydrothermodynamics. For example, for the n-point probability density $f_n = p_{M_1 \ldots M_n}(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ of the velocity field we obtain the equation

$$(1)\quad \begin{cases} \dfrac{\partial f_n}{\partial t_l} = -\left(\dfrac{\partial u_{l\alpha} f_n}{\partial x_{l\alpha}} + \dfrac{\partial w_{l\alpha} f_n}{\partial u_{l\alpha}}\right), \\ w_{l\alpha} = E(\mathbf{x}_l, \mathbf{x}) \left\langle -\dfrac{1}{\rho_0}\dfrac{\partial p}{\partial x_\alpha} + \nu \Delta u_\alpha - g\dfrac{\rho}{\rho_0}\delta_{\alpha 3} + 2e_{\alpha\beta\gamma}\omega_\gamma u_\beta \right\rangle, \end{cases}$$

where the Greek indices denote the numbers of the Cartesian components ($\alpha = 3$ is the vertical component with the positive direction upwards), and repeated Greek indices are to be summed. The angular brackets denote the conditional mathematical expectations of the random values of the hydrothermodynamic fields at the point $(\mathbf{x}, t_k)$ under the condition that the values $\mathbf{u}_1 = \mathbf{u}(M_1), \ldots, \mathbf{u}_n = \mathbf{u}(M_n)$ are fixed. The expression in the angular brackets is the hydrodynamic acceleration, written in the Boussinesq approximation usually used in geophysics, in which the deviations of the pressure p and the density ρ from their standard hydrostatic vertical distributions p_0 and ρ_0 are assumed to be small, and the equations are linearized with respect to them, after which the equation of continuity is taken as a condition for the velocity field to be solenoidal (ν is the kinematic coefficient of molecular viscosity, the third term is the Archimedean acceleration, g is the acceleration of gravity, $\delta_{\alpha\beta}$ is the identity tensor, the fourth term is the Coriolis acceleration, ω_γ is the angular velocity vector of the planet, $e_{\alpha\beta\gamma}$ is the antisymmetric unit tensor, and the specific nature of the geophysical hydrodynamics is due to the third and fourth terms). Finally, $E(\mathbf{x}_l, \mathbf{x})$ is the operator replacing $\mathbf{x}$ by $\mathbf{x}_l$.

The quantities $w_{l\alpha}$ can be expressed in terms of the n- and $(n+1)$-point probability distributions f_n and f_{n+1}, so the equations (1) form an infinite chain (completely analogous to the chain of Bogolyubov equations for the n-particle distribution functions of the velocities of molecules in the kinetic theory of gases). On the other hand, these equations are linear in the functions $f_1, \ldots, f_n, \ldots$, hence, unlike the hydrodynamics of individual realizations of flows of fluids and gases, which is non-linear, the statistical hydrodynamics of turbulent flows turns out to be a linear problem. Multiplying (1) by $F(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ and integrating the result over all $\mathbf{u}_1, \ldots, \mathbf{u}_n$, we obtain

$$(2)\quad \frac{\partial \overline{F}}{\partial t_l} = -\frac{\partial \overline{u_{l\alpha} F}}{\partial x_{l\alpha}} + \overline{w_{l\alpha}\frac{\partial F}{\partial u_{l\alpha}}},$$

where the overbar denotes the mathematical expectation. For $F = \prod_{l=1}^{n} [u_{j_l}(M_l)]^{m_l}$ the relations (2) form an infinite system of linear equations for all possible n-point statistical moments $\overline{F}$ of the velocity field with order $m_1 + \ldots + m_n$ (this system was first derived in 1924 by Friedmann and Keller [2]). If it is assumed that all the finite-dimensional probability distributions for the hydrothermodynamic fields of a turbulent flow are completely determined by their moments, then the Friedmann-Keller system of equations gives a complete description of the statistical dynamics of turbulence, as does the more compact system (1). But the most compact description is given in terms of the characteristic functional defined for the velocity field $\mathbf{u}(M)$ by the formula

$$\Phi[\theta(M)] = \overline{\exp\left[i \int \theta_\alpha(M)\, u_\alpha(M)\, dM\right]}, \tag{3}$$

which has the meaning of the functional Fourier transformation of the probability measure on the function space of dynamically possible velocity fields, and whose values on linear combinations $\sum_{l=1}^{n} \theta_l \delta(M - M_l)$ of delta-functions are the characteristic functions of the finite-dimensional probability distributions. A linear equation in the variational derivatives that is equivalent to the systems of equations (1) or (2) can be derived for the characteristic functional from the equations of hydrodynamics. In particular, for the time evolution of the spatial characteristic functional of the velocity field of an incompressible fluid

$$\Phi[\theta(\mathbf{x}),\ t] = \overline{\exp\left[i \int \theta_\alpha(\mathbf{x})\, u_\alpha(\mathbf{x},\ t)\, dx\right]}\ ; \tag{4}$$

this equation was derived in 1952 by Hopf [3]. It turned out to be completely analogous to the Schrödinger equation for the state vector of a Bose quantum field with strong interactions, except with the stipulation that [illegible]

interactions (see §28 in the book [1], and the article [4]). A number of existence and uniqueness theorems for solutions of the equations for the characteristic functionals in statistical hydromechanics are proved in the recent book of Vishik and Fursikov [5] by the method of constructing probability measures for the successive Galerkin approximations of the equations of hydrodynamics (weak compactness is proved for the sequence of such measures). However, it has not yet come to the application of this method to actual turbulent flows and especially not to geophysical flows, which are complicated by Archimedean forces and by rotation.

An intermediate (with regard to analytic inconvenience) formulation of the turbulence problem in terms of finite-dimensional probability distributions has "not worked out" historically: the equations (1) were introduced only in 1967 (in [6] and [7], and also by Lundgren [8] for $n = 1$ and 2), and convincing ways of obtaining from them some finite system of equations (similar, for example, to the derivation of the Boltzmann equation in the kinetic theory of gases) have not yet been worked out (we mention here only the proposals of Ulinich and Lyubimov [9] and Ievlev [10]).

On the other hand, many diverse constructive ways of obtaining approximate finite systems of equations have been developed for the

Friedmann-Keller equations (2), and it is these equations, along with certain similarity hypotheses, that have been used and are being used up to the present time to compute a number of statistical characteristics of actual turbulent flows, including geophysical flows (we note, however, that practically all these ways—as is the case in the kinetic theory of gases—lead to non-linear equations, thereby violating the principle of linearity of statistical dynamics and the consequent possibility of linear superposition of probability measures; this is an "atonement" for not taking into account the fact that in statistically non-equilibrium states with non-Gaussian probability measures no finite set of moments or even of finite-dimensional probability distributions evolves independently).

§2. The equations for the first and second moments

It is expedient to begin with the simplest of the Friedmann-Keller equations, that is, with the equations (2) for the moments $\overline{F}$ of first and second orders. The equations for the first moments are obtained by a simple averaging of the equations of hydrodynamics (which will be written everywhere in the Boussinesq approximation) and differ from the same equations for the unaveraged hydrodynamic fields by the appearance of additional unknowns: the one-point second moments $\tau_{jl} = -\overline{\rho_0 u'_j u'_l}$ having the meaning of turbulent flows of momentum (the so-called Reynolds stresses), the masses $M_j = \overline{\rho' u'_l}$, the entropy, the heat, and thermodynamically active admixtures (here and below, the primes denote turbulent fluctuations of hydrodynamic fields, that is, their deviations from their averaged values). In the so-called semi-empirical theories these turbulent flows are described, by analogy with the corresponding molecular flows, as linear functions of the gradients of the averaged hydrodynamic fields; the coefficients of these linear functions have the meanings of the coefficients of turbulent viscosity, heat conductivity, and diffusion. To determine them in the theories of Taylor (1915, 1932) and Prandtl (1925) the concept is introduced of the "mixing path" l, which is analogous to the mean free path of molecules in the kinetic theory of gases and, for example, in the case (often encountered in geophysics) when the average flow is horizontal, homogeneous along the horizontal, and plane parallel, so that for $(x_1, x_2, x_3) = (x, y, z)$ and $(u_1, u_2, u_3) = (u, v, w)$ we have $\overline{u} = \overline{u}(z)$ and $\overline{v} = \overline{w} = 0$, w' is set equal to $l'\partial\overline{u}/\partial z$, and the kinematic coefficient of turbulent viscosity is defined by the formula

$$K = \overline{l'w'} = l^2 \left| \frac{\partial \overline{u}}{\partial z} \right|, \tag{5}$$

where $l = (\overline{l'^2})^{1/2}$ is the so-called turbulence scale. Finally, Kármán's theory (1930) used the hypothesis of local kinematic similarity of turbulence (Kármán himself did not formulate the hypothesis realistically, that is, for individual realizations of turbulent flows; it was capable of making sense only for the average characteristics), which yields the relation

$$l = -\varkappa \frac{\partial \bar{u}}{\partial z} \left(\frac{\partial^2 \bar{u}}{\partial z^2} \right)^{-1}, \tag{6}$$

where $\varkappa$ is a numerical constant (which turned out to be close to 0.4).

Taylor (1935) introduced the concept of isotropic turbulence, in which all the finite-dimensional probability distributions f_n are invariant under any translations, rotations, and reflections in the space of vectors $\mathbf{x}$. This is the geometrically simplest case, but real turbulent flows do not have such properties, and even approximately isotropic turbulence can be created only artificially—behind lattices in wind tunnels. With time (or with distance from the lattice) such turbulence degenerates. According to the Kármán hypothesis, this degeneration is self-similar, that is, when the lengths and velocities are measured by certain scales $l(t)$ and $v(t)$ (for example, $v(t) = [b(t)]^{1/2}$, where b is the kinetic energy of the turbulence per unit mass), it is described by certain universal functions. True, experimental data show that there is only incomplete self-similarity, with power dependence on the dimensionless time $(t-t_0)U/L$, where L is the size of the cells in the lattice and U is the velocity of the flow impinging on it [11]. But if Re is large, then for the two-point moments with the distance $r = |\mathbf{x}_2 - \mathbf{x}_1|$ between the points not too small it is possible to prove approximate self-similarity, and in one particular case of this (for a finite so-called Loitsyanskii integral) Kolmogorov [12] obtained the following laws of growth of the scale and of degeneration of the energy for developed isotropic turbulence:

$$\ldots)^{2/7} \qquad \ldots \left(10 \ldots \right)^{2} \ldots)^{-10/7}$$

where c is a dimension constant.

Kolmogorov introduced a number of fundamental new ideas in the theory of turbulence. The most important of these were his ideas (1941) about local isotropy for any turbulence with a very large Reynolds number $\mathrm{Re} = \nu^{-1}LU$ (where L and U are the length and velocity scales typical for the flow, and ν, as in (1), is the kinematic coefficient of molecular viscosity), that is, about the invariance of the finite-dimensional probability distributions for the relative velocities $\mathbf{v}(M) = \mathbf{u}(M) - \mathbf{u}(M_0)$ (for fixed $\mathbf{u}_0 = \mathbf{u}(M_0)$) under arbitrary translations in the space of points M and any rotations and reflections in the space of vectors $\mathbf{x}$, and his similarity hypotheses for very small-scale components in locally isotropic turbulence ([13], [14]). This theory has been strikingly confirmed in countless experimental data (see, for example, §23 in [1], and [15]) and has emerged as one of the greatest discoveries in hydromechanics during the 20th century. In geophysics it has especially broad applications to the atmosphere, in which the Reynolds numbers are, as a rule, very large. However, we shall not present this theory here, referring the readers to Obukhov's article in this same issue, but instead we shall dwell on Kolmogorov's 1942 article [16], which deals with a description of very large-scale components of turbulence (the components which make the main contribution to the transfer of momentum, heat, and admixtures) and has given rise to a broad direction of investigations in the theory of turbulence.

In this article Kolmogorov proposed firstly, by analogy with self-similar isotropic turbulence, to express the one-point characteristics of any turbulence that is not too anisotropic in terms of its energy b and scale l, so that, for example, the kinematic coefficient of turbulent viscosity K and the average rate of dissipation of turbulent energy per unit mass of fluid ϵ must be expressed by the formulae

$$K \sim b^{1/2} l, \quad \varepsilon \sim b^{3/2} l^{-1}. \tag{8}$$

Secondly, he proposed to determine b by using the equation of turbulent energy derived from the equations of hydrodynamics; to within certain small terms it has the form

$$\frac{\partial b}{\partial t} = -\overline{u'_\alpha u'_\beta}\frac{\partial \bar{u}_\beta}{\partial x_\alpha} - \frac{g}{\rho_0} M_z - \varepsilon - \frac{\partial}{\partial x_\alpha}(\bar{u}_\alpha b - \overline{u'_\alpha b'}), \tag{9}$$

where $b' = (1/2)u'_\beta u'_\beta$ (see [17] for a detailed derivation of this equation and its use in computing the turbulence in the boundary layer of the atmosphere). In the case of a horizontal and horizontally homogeneous average flow we can introduce in (9) the coefficients of turbulent viscosity K and diffusion K_1 according to the rules of the semi-empirical theory of turbulence and get

$$\frac{\partial b}{\partial t} = K\left[\left(\frac{\partial \bar{u}}{\partial z}\right)^2 + \left(\frac{\partial \bar{v}}{\partial z}\right)^2\right](1 - \mathrm{Rf}) - \varepsilon + \frac{\partial}{\partial z} K_1 \frac{\partial b}{\partial z}, \tag{10}$$

$$\mathrm{Rf} = -\frac{g}{\rho} M_z \left[\overline{u'w'}\frac{\partial \bar{u}}{\partial z} + \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right]^{-1},$$

where Rf is the so-called dynamical Richardson number (the ratio of the work done by Archimedean forces to the work done by Reynolds stresses). Finally, for determining the scale of turbulence l Kolmogorov proposed using the equation for the typical frequency of turbulent pulsations $\omega = b^{1/2}l^{-1}$, which in the case of an isotropic turbulence degenerating according to the self-similar law (7) has the form $\partial\omega/\partial t = -0.7\omega^2$, while for anisotropic turbulence a diffusion term of the type $(\partial/\partial z)K_1(\partial\omega/\partial z)$ analogous to the last term in (10) must be added. However, this suggestion is not definitive, because the self-similar degeneration of isotropic turbulence (7) is only a partial solution; experiments testify only to incomplete self-similarity. In this connection, no definitive complementary equation for determining l that is as widely accepted as (10) has yet been obtained, and l is usually specified on the basis of various additional arguments, for example, Kármán's formula (6) or its generalization

$$l = -\varkappa\Psi\left(\frac{\partial\Psi}{\partial z}\right)^{-1}, \quad \Psi^2 = \left[\left(\frac{\partial \bar{u}}{\partial z}\right)^2 + \left(\frac{\partial \bar{v}}{\partial z}\right)^2\right](1 - \mathrm{Rf}). \tag{11}$$

The equation (10) has found numerous applications in calculations of actual turbulent flows, in particular, in geophysics. We remark that for $K_1 \sim b^{1/2}l$ the diffusion of turbulence described by the last term in (10) has a non-linear character, and this gives rise to specific effects: for diffusion into a layer at rest with neutral stratification of the density (M_z = Rf = 0) the turbulence (the value of b) fades out with the distance according to a power law, while for diffusion into a layer with stable stratification ($M_z > 0$, Rf > 0) it can penetrate only to a finite distance.

We give an illustration of a use of (10) indicated by Kolmogorov and not published up to now. Consider a steady average flow parallel to the x-axis ($\bar{v} = 0$) in a homogeneous fluid ($M_z = 0$) with equation of motion $K\partial\bar{u}/\partial z = \tau$, so that all its characteristics can be expressed in terms of τ and $\partial\bar{u}/\partial z$. In particular, $K = \tau(\partial\bar{u}/\partial z)^{-1}$ and $b \sim \tau$, so (10) takes the form

$$0 = \tau\frac{\partial \bar{u}}{\partial z} - \varepsilon + c\frac{\partial}{\partial z}\tau\left(\frac{\partial \bar{u}}{\partial z}\right)^{-1}\frac{\partial \tau}{\partial z}. \tag{12}$$

In the first approximation $\varepsilon \approx \varepsilon_0 = \alpha\tau\,\partial\bar{u}/\partial z$, and in the second approximation we should take into account the "dissipation flow" $\Pi = \beta\tau(\partial\bar{u}/\partial z)^{-1}(\partial\epsilon_0/\partial z)$ calculated according to the first approximation, and add to ϵ_0 the divergence $\partial\Pi/\partial z$ of this flow, provided with the factor $(\partial\bar{u}/\partial z)^{-1}$ from dimensional considerations. Confining ourselves below to the case τ = const, in which the last term on the right-hand side of (12) vanishes, we reduce (12) to the form

$$\tau\left(\frac{\partial \bar{u}}{\partial z}\right)^2 = a\frac{\partial}{\partial z}\tau\left(\frac{\partial \bar{u}}{\partial z}\right)^{-1}\frac{\partial}{\partial z}\tau\frac{\partial \bar{u}}{\partial z}, \quad a = \frac{\beta}{1-\alpha}, \tag{13}$$

from which

(14) [illegible]

By the substitution $x = \log F$ and $y = (\partial x/\partial\zeta)^2$ the equation for $F(\zeta)$ can be brought to the form $\partial y/\partial x = 2e^{2x}$, from which $y = c + e^{2x}$, so that $\partial x/\partial\zeta = (c+e^{2x})^{1/2}$ and, consequently,

$$\zeta = \int (c + e^{2x})^{-1/2}\partial x. \tag{15}$$

It suffices to give the constant c the values 0 and ± 1. Inverting the dependence $\zeta(x) = \zeta(\log F)$, we obtain three forms of solutions: $F = \zeta^{-1}$, $(\cos\zeta)^{-1}$, and $(\sinh\zeta)^{-1}$. The first of these leads to a logarithmic law for the profile of the velocity $\bar{u}(z)$ (which is characteristic for turbulent flows near a wall), the second leads to a Couette flow between two moving walls, and the third $\bar{u}(z)$ has a logarithmic character for small z and approaches a constant asymptotically for large z.

A description of real turbulence analogous to (9) but more detailed is given by the evolution equations derived from the equations of hydrodynamics for all the one-point second moments of the turbulent pulsations of the hydrothermodynamic fields. We write these equations out here in covariant form, having in view possibilities for their use in geophysics in both

Cartesian and spherical coordinates. For the second moments $b_{ij} = \overline{u_i' u_j'}$ of the velocity field the equations have the form

$$(16)\quad Db_{ij} + (b_{i\alpha}\nabla^{\alpha}\bar{u}_j + b_{j\alpha}\nabla^{\alpha}\bar{u}_i) + (\overline{f_i' u_j'} + \overline{f_j' u_i'}) - \\ - \frac{g}{\rho_0}(b_{i\rho}\lambda_j + b_{j\rho}\lambda_i) = -\frac{1}{\rho_0}(\overline{u_j'\nabla_i p'} + \overline{u_i'\nabla_j p'}) + \\ + \frac{1}{\rho_0}(\overline{u_j'\nabla^{\alpha}\tau_{i\alpha}'} + \overline{u_i'\nabla^{\alpha}\tau_{j\alpha}'}) \approx -c_1 b^{1/2} l^{-1}\left(b_{ij} - \frac{2}{3} b g_{ij}\right) - \\ - c_2 b^{3/2} l^{-1}\left(\lambda_i\lambda_j - \frac{1}{3} g_{ij}\right) - 2b^{3/2} l^{-1}[(c_3 - 3c_4)\lambda_i\lambda_j + c_4 g_{ij}],$$

where $D = \partial/\partial t + \bar{u}_\alpha\nabla^\alpha - \nabla_\alpha K_1 \nabla^\alpha$, the f_i' are the pulsations of the Coriolis acceleration, $b_{i\rho} = \overline{u_i'\rho'}$, λ_i is the unit vector in the vertical direction, the τ_{ij}' are the pulsations of the molecular stresses, g_{ij} is the metric tensor, and c_1, c_2, c_3, and c_4 are numerical constants. The middle part of the equality involves the correlations between the velocity pulsations and the stress pulsations taken at the same point, and the right-hand part gives semi-empirical expressions for these correlations in the spirit of Rotta [18]: the first of these terms describes the correlations between the velocities and the pressure pulsations, the second the correlations between the velocities and the viscous stresses (the anisotropic "dissipations" of the moments b_{ij}), and the third the anisotropy created by the presence of a horizontal boundary $z = 0$ for the flow. In the case of the Earth's atmosphere $\rho'/\rho_0 \approx -T'/T_0$, where T is the temperature, and the equations (16) must be complemented by analogous evolutionary equations for the moments $b_{iT} = \overline{u_i'T'}$, which are obtained in the form

$$(17)\quad Db_{iT} + b_{\alpha T}\nabla^{\alpha}\bar{u}_i + b_{i\alpha}\nabla^{\alpha}\bar{\theta} + \overline{f_i'T'} - g b_{TT}\lambda_i = \\ = -\frac{1}{\rho_0}\overline{T'\nabla_i p'} + \frac{1}{\rho_0}\overline{T'\nabla^{\alpha}\tau_{i\alpha}'} + \frac{1}{\rho_0}\overline{u_i'\nabla^{\alpha}\rho_0\chi\nabla_\alpha T'} \approx \\ \approx -c_5 b^{1/2} l^{-1} b_{iT} - c_6 b l^{-1} b_{TT}^{1/2}\lambda_i - c_7 b l^{-1} b_{TT}^{1/2}\lambda_i,$$

where θ is the potential temperature, χ is the coefficient of molecular heat conductivity, and c_5, c_6, and c_7 are new numerical constants. Finally, the evolutionary equation

$$(18)\quad Db_{TT} + b_{\alpha T}\nabla^{\alpha}\bar{\theta} = \frac{1}{\rho_0}\nabla^{\alpha}\rho_0\chi\overline{T'\nabla_\alpha T'} - \chi\overline{\nabla_\alpha T'\nabla^{\alpha}T'} \approx -c_8 b^{1/2} l^{-1} b_{TT}$$

is obtained for the moment $b_{TT} = \overline{T'^2}$ appearing in (17), where c_8 is another constant. The equations (16)-(18) (with slight simplifications) were used in [19]-[21] to compute the characteristics of turbulence in the surface and boundary layers of the Earth's atmosphere. Analogous equations were used in [22]-[24] to compute the characteristics of turbulence in the convective layer of the Sun for the purpose of explaining its differential rotation without invoking the hypothesis of "anisotropic viscosity". Finally, $\rho'/\rho_0 \approx -\alpha T' + \beta S'$ in the case of the ocean, where S is the salinity, and the equations (17) and (18) must be complemented by analogous equations for the moments b_{LS}, b_{SS}, and b_{ST} (see [25]).

In each of these three cases—the Earth's atmosphere, the convective layer of the Sun, and the ocean—a closed system of semi-empirical equations is obtained for the one-point first and second moments of the hydro-thermodynamic fields, provided only that the scale of turbulence l appearing in them is specified somehow. The simplest case (considered in [19]) is that of a steady-state horizontally homogeneous (SSHH) near-surface layer of the atmosphere. In this case, when the diffusion terms in D are ignored and the density stratification is neutral, it is possible to express the numerical constants on the right-hand sides of (16)–(18) in terms of directly measurable quantities; these numerical constants can then be used in more complicated cases.

§3. Turbulence in stratified media

Kolmogorov initiated the investigation of turbulence in stratified media under conditions of stable density stratification, when $M_z > 0$ (and $\mathrm{Rf} > 0$) and, as is clear from the energy equation (9) or (10), the turbulence loses energy by work against the Archimedean forces of buoyancy, and also under conditions of unstable stratification, when $M_z < 0$ (and $\mathrm{Rf} < 0$) and the turbulence, conversely, receives energy (in the form of convective motions) due to the work of the Archimedean forces. These effects of stratification are one of the specific features of geophysical hydrodynamics.

Turbulence in stratified SSHH-flows (except for its very small-scale components, whose mode is essentially influenced by the action of molecular viscosity, heat conductivity, and diffusion) can be described with the help of [illegible] with the near-surface layer of the atmosphere by Obukhov [26] and Monin [27]. After these authors confirmed it in joint work ([28], [29]) with data from measurements of velocity profiles $\overline{u}(z)$ of a vortex and measurements of the temeprature $\overline{T}(z)$ in the near-surface layer of air, this similarity theory became widely used as a basis for a quantitative description of the characteristics of atmospheric tubulence (see Ch. IV in [1], and the book [30] by Lamley and Panofsky). The theory was then extended to the boundary layer of the atmosphere, in the dynamics of which the Coriolis force due to the rotation of the Earth takes on essential significance ([31]–[32], [21]), and in this form it turned out to be applicable also to the upper layer of the ocean [25]. We present it here in this last, most general version.

According to this similarity theory, the characteristics of the turbulence in a stratified layer spanned by an SSHH-flow are completely described by five "external parameters": the thickness h of the layer, the "height of the roughness" of its boundary z_0, the "buoyancy parameter" g/ρ_0, and the falls in the velocity U and the potential density $\delta\rho_*$ across the layer. But if it is also necessary to determine the characteristics of the turbulent fields of temperature and salinity, then instead of $\delta\rho_*$ it is necessary to give the falls

δT and δS in temperature and salinity across the layer (then $\delta\rho_* \approx -\rho_0\alpha\delta T + \rho_0\beta\delta S$) and instead of g/ρ_0 the parameters αg and βg, where α and β are the coefficients of compressibility of sea water mentioned above.

These "external parameters" determine, in particular, such important "internal parameters" as the "rate of friction" $u_* = (\tau/\rho_0)^{1/2}$ (where τ is the frictional stress) on the rough boundary of the layer and the vertical turbulent flows of mass $M_z = \overline{\rho' w'}$, heat $q_z = C_p\rho_0\overline{T'w'}$ (C_p is the specific heat for a constant pressure), and salt $I_z = \rho_0\overline{S'w'}$ on this boundary (moreover, $M_z = -(\alpha/C_p)q_z + \beta I_z$), and thus also the thickness $|L|$ of a sublayer with turbulent flows constant with respect to z, and the scales of variations of the density R_*, temperature T_*, and salinity S_*, which are given by the formulae

$$(19)\qquad \begin{cases} L = u_*^3\left(\varkappa \dfrac{g}{\rho_0} M_z\right)^{-1}, & R_* = M_z(\varkappa u_*)^{-1}, \\ T_* = q_z(\varkappa C_p\rho_0 u_*)^{-1}, & S_* = I_z(\varkappa\rho_0 u_*)^{-1} \end{cases}$$

(where $\varkappa$ is the numerical "Kármán constant" that appeared in (6) and is introduced here for convenience). Especially important is the scale of thickness L, which was introduced in [26]-[29] and characterizes the stratification. For a stable density stratification we have $M_Z > 0$, $L > 0$, and $R_* > 0$, and for unstable stratification we have, conversely, $M_z < 0$, $L < 0$, and $R_* < 0$. The same "external parameters" also determine the angle γ of rotation of the flow across the layer, the "coefficient of friction" $\eta = u_*/U$, and the "internal parameter of stratification" $\mu = h/L$, which was first introduced in [31], [32]. According to this similarity theory the one-point probability density of the values $\mathbf{u}$, ρ, T, and S must have the form

$$(20)\quad p(\mathbf{u}, \rho, T, S) = u_*^{-3}|R_* T_* S_*|^{-1}\mathcal{F}\left(\frac{\mathbf{u}}{u_*}, \frac{\rho}{R_*}, \frac{T}{T_*}, \frac{S}{S_*}, \frac{z}{h}, \mu, \frac{z_0}{h}\right),$$

where $\mathcal{F}$ is a universal function, and z is measured from the boundary of the layer. We remark that only the constant terms in the formulae for the average hydrodynamic fields can depend on z_0, for example,

$$(21)\qquad \bar{u}(z) = \bar{u}(z_0) + \frac{u_*}{\varkappa}\left[f_u\left(\frac{z}{h}, \mu\right) - f_u\left(\frac{z_0}{h}, \mu\right)\right],$$

but the vertical gradients of the average fields and the probability distribution for the pulsations $\mathbf{u}'$, ρ', T', and S' must not depend on z_0. For $z/h \ll 1$ the dependence on h in (20) must disappear, that is, the function $\mathcal{F}$ becomes dependent not on the three arguments z/h, μ, and z_0/h, but only on two of their products $\zeta = \mu z/h = z/L$ and $\zeta_0 = \mu z_0/h = z_0/L$; in particular, since $z_0/h \ll 1$, we can set

$$f_u\left(\frac{z_0}{h}, \mu\right) \approx f_u\left(\frac{z_0}{L}\right)$$

in (21).

Consider the condition $z \ll |L|$. It can be realized both by a decrease in z and as an increase in $|L|$ by decreasing $|M_z|$, that is, by approximating the neutral stratification. Thus, the small z form a sublayer with neutral stratification "close to the wall". In it the influence of the parameter μ must disappear, so that instead of the three arguments z/h, μ, and z_0/h or the pair z/L and z_0/L of their products there can only be a dependence on the ratio z/z_0 remaining. In this sublayer asymptotic expressions of the following form are obtained from formulae of the type (21) for the vertical gradients of the average fields:

$$\frac{\partial \bar{u}}{\partial z} = \frac{u_*}{\varkappa h} f_u'\left(\frac{z}{h}, \mu\right) \sim \frac{u_*}{\varkappa z} \varphi_u\left(\frac{z}{L}\right) \approx \frac{u_*}{\varkappa z}, \tag{22}$$

if $\varphi_u(0)$ is set equal to 1, which is equivalent to the determination of the constant $\varkappa$. Thus, the asymptotic behaviour "close to the wall" turns out to be logarithmic for formulae of the type (21). At the other boundary h of the layer, formulae of the type (21) must yield the so-called "laws of defect", for example, $\bar{u}(z) = (u_*/h)\psi_u(z/h, \mu)$. For small z they must be satisfied simultaneously with the logarithmic laws "close to the walls", so by equating them and introducing for the dimensionless quantities independent of z notation of the type

$$\frac{\varkappa U \cos\gamma}{u_*} - \log\frac{z_0}{h} = \psi_u\left(\frac{z}{h}, \mu\right) - \log\frac{z}{h} \to A(\mu) \quad \left(\frac{z}{h} \to 0\right), \tag{23}$$

we can derive the following laws of drag, mass exchange, heat exchange, and salt exchange:

$$\left\{\begin{aligned} &\frac{U}{u_*} = \frac{1}{\varkappa}\left\{B^2(\mu) + \left[A(\mu) - \log\frac{h}{z_0}\right]^2\right\}^{1/2}, \quad \sin\gamma = \frac{1}{\varkappa}\frac{u_*}{U} B(\mu), \\ &\frac{\delta\rho_*}{R_*} = \frac{1}{\alpha_\rho}\left[C(\mu) + \log\frac{h}{z_0}\right], \quad \frac{\delta T}{T_*} = \frac{1}{\alpha_T}\left[D(\mu) + \log\frac{h}{z_0}\right], \\ &\frac{\delta S}{S_*} = \frac{1}{\alpha_S}\left[E(\mu) + \log\frac{h}{z_0}\right]. \end{aligned}\right. \tag{24}$$

Finally, the "internal parameter of stratification" μ can be expressed in terms of the "external Richardson number" Ri and other external parameters with the help of the relation

$$\mathrm{Ri} = \frac{gh\delta\rho_*}{\rho_0 U^2} = \frac{\mu}{\alpha_\rho}\left[C(\mu) + \log\frac{h}{z_0}\right]\left\{B^2(\mu) + \left[A(\mu) - \log\frac{h}{z_0}\right]^2\right\}^{-1}. \tag{25}$$

The formulae (24) and (25) enable us to express the characteristics of the turbulence in a stratified SSHH-flow in terms of its "external parameters". A particular case of such a flow important for geophysics is the so-called Ekman boundary layer (EBL), for which $h = h_0\Phi(\mu_0)$, where $h_0 = \varkappa u_*/f$ is the thickness of the neutrally stratified EBL (f is the Coriolis parameter), $\mu_0 = h_0/L$, and $\Phi(\mu_0)$ is some universal function. In this case the whole similarity theory presented above remains in force when h and μ are replaced by h_0 and μ_0; here $h_0/z_0 = (\varkappa u_*/U)\mathrm{Ro}$ in (24), where $\mathrm{Ro} = U/(z_0 f)$ is the so-called Rossby number, which is determined by only external parameters (it was introduced by Rossby as far back as 1932).

If $z/h \ll 1$, so that the dependency on h in the formula (20) disappears, all universal functions of $\zeta = z/L$ have intermediate asymptotic behaviour for strongly unstable ($\zeta \to -\infty$) and strongly stable ($\zeta \to +\infty$) stratifications, which most strikingly demonstrates the influence of stratification on turbulence. For example, values $\zeta \to -\infty$ can be attained for fixed z and $M_z < 0$ when $u_* \to 0$, that is, the case of free convection (with the absence of an average flow) is asymptotic for a strongly unstable stratification. In this case not only the parameter h but also u_* must drop out of the formulae (20) and (21), and, since it is not possible to form a constant scale of length from the remaining parameters g/ρ_0 and M_z, the free convection mode turns out to be self-similar—the universal functions of ζ become power functions. Here the formulae of the type (21) take the form

$$(26)\qquad \begin{cases} \bar{u}(z_2)-\bar{u}(z_1) = -\dfrac{Cu_*}{\varkappa}\left(|\zeta_2|^{-1/3}-|\zeta_1|^{-1/3}\right), \\ \bar{\rho}_*(z_2)-\bar{\rho}_*(z_1) = \dfrac{C|R_*|}{\alpha_\rho(-\infty)}\left(|\zeta_2|^{-1/3}-|\zeta_1|^{-1/3}\right), \end{cases}$$

where C and $\alpha_\rho(-\infty)$ are certain numerical constants (formulae analogous to the second of these are valid for $\overline{T}(z)$ and $\overline{S}(z)$), and the probability density for the pulsations $\mathbf{u}'$, ρ', T', and S' takes the form

$$(27)\quad p(\mathbf{u}', \rho', T', S') \sim \sim u_*^{-3}|R_* T_* S_*|^{-1} \mathscr{F}_1\left(\frac{\mathbf{u}'}{u_*|\zeta|^{1/3}}, \frac{\rho'}{R_*|\zeta|^{-1/3}}, \frac{T'}{T_*|\zeta|^{-1/3}}, \frac{S'}{S_*|\zeta|^{-1/3}}\right)$$

(concerning this asymptotic expression see [33] by Kazanskii and Monin and [34] by Obukhov). For a strongly stable stratification ($\zeta \to +\infty$) the dynamical Richardson number Rf in the second formula in (10) tends to some limit value R, hence, the gradient $\partial\bar{u}/\partial z$ of the average velocity becomes constant:

$$(28)\qquad \mathrm{Rf} = gM_z\left(\rho u_*^2 \frac{\partial\bar{u}}{\partial z}\right)^{-1} \to R, \quad \frac{\partial\bar{u}}{\partial z} \sim \frac{gM_z}{R\rho_0 u_*^2} = \frac{u_*}{\varkappa RL}$$

(see [35]). The gradients of the average density, temperature, and salinity are given by the asymptotic formulae

$$(29)\qquad \frac{\partial\bar{\rho}_*}{\partial z} \sim \frac{R_*}{\alpha_\rho(\zeta)RL}, \quad \frac{\partial\overline{T}}{\partial z} \sim \frac{T_*}{\alpha_T(\zeta)RL}, \quad \frac{\partial\overline{S}}{\partial z} \sim \frac{S_*}{\alpha_S(\zeta)RL},$$

moreover, $\alpha_\rho \sim \zeta^{-2}$ according to [36], and the same should clearly be expected from the functions α_T and α_S. Finally, the turbulent pulsations $\mathbf{u}'$, ρ', T', and S' take on a local character here, that is, their characteristics cease to depend explicitly on z, so their probability density differs from (20) by the absence of the arguments z/h, μ, and z_0/h.

Thus, the pulsations of velocity (that is, the turbulence proper) increase with a decrease in the stability and an increase in the instability of the stratification. The density pulsations are very small for very strong stability; with a decrease in stability the turbulence becomes stronger, but $\partial\bar{\rho}_*/\partial z$

decreases and the density pulsations grow only until a certain moderate stability is attained, after which they decrease to a minimum for neutral stratification; then with increasing instability they grow rapidly, but for very strong instability $\overline{\rho}_*$ levels off and the growth in the density pulsations slows down (and perhaps stops or is even replaced by a decrease in them). The behaviour of the temperature and salinity pulsations in the ocean can be still more complicated.

Let us consider briefly the influence of stratification on the spectrum of turbulence. We confine ourselves to the so-called inertial-convective interval of the spectrum of wave numbers k. By the similarity hypotheses of Kolmogorov [13], Obukhov ([37], [38]), and Corrsin [39], in this interval the mode of turbulence is completely determined by the rates ϵ, ϵ_T, and ϵ_S of transfer of the specific kinetic energy and the measures of non-homogeneity of the fields T and S along the spectrum. According to Obukhov [40] and Bolgiano ([41], [42]), a buoyancy parameter should be added to them in a stratified medium. Then it is possible to determine a constant "buoyancy scale"

$$L_* = \varepsilon^{5/4} \left| \pm \alpha g (\alpha_T \varepsilon_T)^{1/2} \pm \beta g (\alpha_S \varepsilon_S)^{1/2} \right|^{-1/2} \tag{30}$$

(where the upper signs correspond to positive and the lower signs to negative contributions of the gradients $\partial\overline{T}/\partial z$ and $\partial\overline{S}/\partial z$ to the stable stratification of the density), and if L_* falls in the inertial-convective interval of scales, then the spectral densities of the kinetic energy $E(k)$ and the temperature field $E_T(k)$ and salinity field $E_S(k)$ in this interval can be written in the form

$$\left\{ \begin{aligned} E(k) &= \varepsilon^{2/3} L_*^{5/3} f(kL_*), \\ E_T(k) &= \frac{\varepsilon_T}{\alpha_T} \varepsilon^{-1/3} L_*^{5/3} f_T(kL_*), \\ E_S(k) &= \frac{\varepsilon_S}{\alpha_S} \varepsilon^{-1/3} L_*^{5/3} f_S(kL_*) \end{aligned} \right. \tag{31}$$

(see §5 in [25]), where $f(x)$, $f_T(x)$, and $f_S(x)$ are certain universal functions. For large k, that is, in the very small-scale part of the inertial-convective interval, the Archimedean forces are negligibly small in comparison with the inertial forces, so the "buoyancy scale" L_* must drop out of the formulae (31). For this, the intermediate asymptotic behaviour f, f_T, $f_S \sim x^{-5/3}$ must be valid for large x. All the spectra in (31) turn out to be proportional to $k^{-5/3}$; this is the well-known "five-thirds law" of Obukhov and Corrsin for neutrally stratified media.

Conversely, according to [40]-[42], in the case of stable stratification with $k \lesssim L_*^{-1}$ the energy of turbulence transmitted along the spectrum from the large scales to the small scales (the Kolmogorov cascade process) is expended mainly on work against the Archimedean forces, and only a very slight fraction of it reaches the small scales and is dissipated in heat, so the magnitude of ϵ turns out to be small and must asymptotically drop out of the formulae (31). For this, the asymptotic behaviour $f \sim x^{-11/5}$ and f_T, $f_S \sim x^{-7/5}$ must be valid for small x. The spectra in (31) take the forms $E(k) \sim k^{-11/5}$ and $E_T(k)$, $E_S(k) \sim k^{-7/5}$ here. A semi-empirical computation of the functions in (31) is carried out in [43] (see also §5 in [25]). According to this computation, the spectra in (31) have maxima for $kL_* \sim 1$ in the case of an unstable stratification.

In his addresses Kolmogorov conjectured more than once that, due to the large expenditures of energy on work against the Archimedean forces, the turbulent mixing for a very stable stratification cannot envelop the whole fluid for any length of time and must concentrate itself only in separate turbulent layers or "blini" (pancakes) (the term blini thus entered the English-language literature on turbulence). Because of mixing, the layers made turbulent must be quasi-homogeneous along the vertical and must be separated by very thin sheets with microjumps ("steps") in temperature, salinity, density, and the other thermodynamic parameters of the fluid.

This conjecture received brilliant confirmation in the discovery of a thin-layer vertical microstructure in the ocean (see, for example, the survey [44] and §4 in [25]). The prevalence of this structure in the ocean is explained by the fact that under its upper mixed layer the ocean has a very strongly stratified "pycnocline" or "density-jump layer" several tens of meters thick with typical values $|\partial\bar{\rho}/\partial z| \sim 10^{-6}$ g/cm^4 (whereas $|\partial\bar{\rho}/\partial z| \sim 4.10^{-8}$ g/cm^4 in the strongest surface temperature inversions in the atmosphere). The stratification of the ocean pycnocline can be characterized by the value of the so-called Väisälä-Brunt frequency $N = (-g(\partial \log \bar{\rho}_*/\partial z))^{1/2} \sim 3.10^{-2}$ s^{-1}, whereas $N \sim 10^{-2}$ s^{-1} in the lower 10-kilometer layer of the atmosphere. The vertical microstructure of the ocean was revealed as a result of continual measurement of vertical profiles of temperature, electrical conductivity, and speed of sound by hydrophysical probes dropped on cables (instead of the previously practiced measurements in samples of water taken at several different depths by Nansen bathometers).

The turbulent layers in the ocean have thicknesses of order $h \sim 10^3$ to 10^0 cm, and the turbulence in them is weak (incapable of destroying these layers), has a local character (does not depend directly on the depth), and is characterized by relatively small Reynolds numbers $\nu^{-1}hU \sim 10^2$ to 10^5. The mechanism for the general formation of turbulent layers may be horizontal expansion of lens-shaped spots of turbulence ("flying saucers") that are formed when internal gravitational waves break, and that collapse along the vertical under the action of Archimedean forces in a medium with a stable vertical density gradient (see the calculations of the expansion of the spots in [45] and [46]).

Such a mechanism for the formation of the vertical microstructure corresponds to the following non-Kolmogorov cascade process for transmission of energy along the spectrum of scales. A long internal wave (of the first mode) is formed on the primary pycnocline, large spots of turbulence arise when it breaks, and the expansion of these spots creates secondary pycnoclines. Shorter internal waves are formed on them, spots of turbulence not as large arise when these waves break, the spreading of the latter spots creates tertiary pycnoclines, and so on. As a result, a set of primary, secondary, tertiary, and so on, pycnoclines is formed with a corresponding multi-modal spectrum of internal waves, and the appearance of self-similarity can be expected in the very small-scale part. The creation of a corresponding quantitative theory of self-similarity is one of the immediate problems in the statistical hydromechanics of the ocean.

§4. Two-dimensional turbulence

Certain mechanical processes take place differently in two-dimensional space and three-dimensional space (and, in general, differently in even-dimensional and odd-dimensional spaces); for example, the propagation of waves in three-dimensional space satisfies the so-called Huygens principle, that is, localized wave signals keep their localness under propagation, while waves are formed after them in two-dimensional space. This difference makes itself evident also in hydromechanics, where the equations of motion are of hyperbolic type. We shall show that it manifests itself also in statistical hydromechanics: in two-dimensional turbulence the Kolmogorov cascade process of spectral transfer of energy takes place in the reverse direction, from small scales to large scales, and non-local interactions (between components of turbulence with wave numbers remote from each other) make an essential contribution to it; see the survey in [47], §13 in

Accordingly, we consider a two-dimensional flow of an incompressible fluid with velocity field $u = -\partial\psi/\partial y$, $v = \partial\psi/\partial x$, where the vorticity $\omega = \partial v/\partial x - \partial u/\partial y = \Delta\psi$ satisfies the equation

$$\frac{\partial\Delta\psi}{\partial t} + \frac{\partial(\psi, \Delta\psi)}{\partial(x, y)} = \nu\Delta\Delta\psi. \tag{32}$$

The average kinetic energy $E = (1/2)\overline{|\nabla\psi|^2}$ and the enstrophy[(1)] $\Omega = (1/2)\overline{|\Delta\psi|^2}$ satisfy the equations

$$\frac{\partial E}{\partial t} = -2\nu\Omega = -\varepsilon, \quad \frac{\partial\Omega}{\partial t} = -\nu\overline{|\nabla\omega|^2} = -\varepsilon_\omega, \tag{33}$$

from which it is clear that for any finite initial values of E and Ω the value of Ω can only decrease with time, and $\epsilon \to 0$ as $\nu \to 0$, hence, for large Re the kinetic energy is approximately constant, and the Kolmogorov cascade transfer of any significant fraction of energy from small to large wave numbers is impossible. At the same time, the cascade transfer of enstrophy

(1)The word "enstrophy", a transliteration from Russian, does not seem to be a standard term in English, but we have retained it here. It is equivalent to "mean-square vorticity". (Translator) from small to large wave numbers and the corresponding zero limit of the quantity ϵ_ω as $\nu \to 0$ are not excluded. The three-dimensional effect of intensification of the vorticity field due to expansion of the vortex filaments in two-dimensional flows of an ideal fluid is replaced by the law of conservation of enstrophy (and also of the integrals of any power of the vorticity—their existence enables us to prove in this case existence and uniqueness theorems for solutions of the equations of hydrodynamics; see Arnol'd's book [50]).

We introduce mean wave numbers for the energy spectrum $E(k, t)$ and the enstrophy spectrum $k^2E(k, t)$:

$$(34) \qquad k_1(t) = E^{-1}\int_0^\infty kE(k, t)\,dk, \quad k_2(t) = \Omega^{-1}\int_0^\infty k^3E(k, t)\,dk.$$

By setting $kE(k) = [kE^{1/2}(k)]\cdot[E^{1/2}(k)]$ and $k^2E(k) = [k^{1/2}E^{1/2}(k)]\cdot[k^{3/2}E^{1/2}(k)]$ and applying the Cauchy-Bunyakovskii-Schwarz inequality to the integrals of these functions it is possible to prove the inequalities [51]

$$(35) \qquad k_1(t)\,k_2(t) \geqslant k_*^2 = \Omega/E \quad k_1(t) \leqslant k_*, \; k_2(t) \geqslant k_*,$$

which show that a transfer of enstrophy in the direction of large k (an increase in k_2) must be accompanied by a transfer of energy in the direction of small k (a decrease in k_1, the "turbulent viscosity", here must be assumed to be negative) and that in cascade processes the energy cannot accumulate in small-scale motions, and the enstrophy cannot accumulate in large-scale motions. If k_1 is computed from the self-similar energy spectrum $E(k, t) = E^{3/2}tf(E^{1/2}kt)$ obtained when $E \approx$ const, then the rate of growth of the turbulence scale k_1^{-1} with time is found to be equal to $\partial k_1^{-1}/\partial t = aU$, where $U = (2E)^{1/2}$ is the mean-square velocity of motion of the fluid, and $a = \int f(x)dx\left(\int xf(x)dx\right)^{-1}$ is a positive constant.

In the inertial-convective interval the energy spectra $E(k)$ and the passive admixture fields $E_\theta(k)$ can now depend not only on the parameters ϵ and ϵ_θ but also on the parameter ϵ_ω, hence, a constant scale of length $L_\omega = (\epsilon/\epsilon_\omega)^{1/2}$ emerges, and dimensional considerations give us

$$(36) \qquad E(k) = \varepsilon^{2/3}k^{-5/3}f_1(kL_\omega), \quad E_\theta(k) = \varepsilon_\theta\varepsilon^{-1/3}k^{-5/3}f_2(kL_\omega).$$

As Kolmogorov observed, in this situation it is natural to expect that one of the parameters ϵ and ϵ_ω must be essential at one end of the inertial interval, and the other must be essential at the other end. By (35), of ϵ and ϵ_ω only the parameter ϵ must be essential at the very large-scale end (small k), so the functions $f_1(x)$ and $f_2(x)$ must become the positive constants $f_1(0)$ and $f_2(0)$, and "five-thirds laws" are obtained for the spectra (36), as in three-dimensional turbulence (but only with the opposite direction of spectral transfer of energy—from small scales to large scales). Then at the very small-scale end of the inertial interval the parameter ϵ, conversely, must drop out of the relations (36), and for this the asymptotic behaviour $f_1(x) \sim x^{-4/3}$ and $f_2(x) \sim x^{2/3}$ must be valid for large x; hence, these relations take the form

$$E(k) = C_1 \varepsilon_\omega^{2/3} k^{-3}, \quad E_\theta(k) = C_2 \varepsilon_\theta \varepsilon_\omega^{-1/3} k^{-1}. \tag{37}$$

The first of them was first derived in [52] and [53], and the second in [54] and [55]. The stated predictions about the connections between the processes of spectral transfer of energy and enstrophy are confirmed by computation of the rates $\Pi(k)$ and $Q(k)$ of spectral transfer of energy and enstrophy from wave numbers less than k to wave numbers greater than k in the so-called diffusion approximation (Leith [56]):

$$\Pi(k) = -ak^{-1} \frac{\partial}{\partial k} k^{3/2} E^{3/2}(k), \quad Q(k) = -ak^3 \frac{\partial}{\partial k} k^{5/2} E^{3/2}(k), \tag{38}$$

where a is a positive numerical constant; therefore, in the case $E(k) \sim k^{-5/3}$ we get $\Pi = \text{const} < 0$ and $Q = 0$, while in the case $E(k) \sim k^{-3}$ we get $Q = \text{const} > 0$ and $\Pi = 0$.

We remark that under the laws (37) the total enstrophy $\int_{k_0}^{k_{\max}} k^2 E(k)dk$ and the measure of inhomogeneity of the passive admixture field $\int_{k_0}^{k_{\max}} E_\theta(k)\, dk$ diverge logarithmically both as $k_0 \to 0$ and as $k_{\max} \to \infty$. To avoid these divergences it is necessary to introduce logarithmic corrections in the spectra (37). For this let

$$\begin{cases} \Pi(k) \sim kE(k)\tau^{-1}(k), \\ \Pi_\theta(k) \sim kE_\theta(k)\tau^{-1}(k), \\ \tau(k) \sim \left[\int_{k_0}^{k} k'^2 E(k')\,dk'\right]^{-1/2}. \end{cases} \tag{39}$$

This makes it clear that the condition $\Pi(k) = \text{const}$ can ensure only the "five-thirds law", and the wave numbers $k' \sim k$ make the main contribution to the integral in (39), which demonstrates the localness of the spectral transfer of energy. The laws (37) would yield $\tau(k) \sim (\log(k/k_1))^{1/2}$, and the quantities $Q(k)$ and $\Pi_\theta(k)$ would turn out to be increasing with k. To ensure the conditions $Q(k) = \text{const}$ and $\Pi_\theta(k) = \text{const}$, (37) must be replaced by

$$E(k) = C_1 \varepsilon_\omega^{2/3} k^{-3} \left(\log \frac{k}{k_0}\right)^{-1/3}, \quad E_\theta(k) = C_2 \varepsilon_\theta \varepsilon_\omega^{-1/3} k^{-1} \left(\log \frac{k}{k_0}\right)^{-1/3}. \tag{40}$$

Here the wave numbers k' in the interval $k \gg k' \gg k_0$ make the main contribution to the integral in (39) when $k \gg k_0$, hence, the spectral transfers of enstrophy and of the passive admixture turn out to be non-local processes (Kraichnan [57]). According to computations of Mirabel' and the author, the coefficient of turbulent diffusion $K(r)$ in the inertial interval of distances r has the form

(41) $$K(r) \sim A\varepsilon_\omega^{1/3} r^2 (\log k_0 r)^{-2/3},$$

where A is a numerical constant. Because of the logarithmic factor arising from non-localness, we see that $r^{-2}K(r) \to 0$ ($r \to \infty$), which ensures the existence of a solution corresponding to an instantaneous point source of the admixture for the semi-empirical equation of turbulent diffusion.

§5. Geostrophic turbulence

In geophysics an equation analogous to (32) can be derived from the equations of evolution of the entropy η and the potential vortex $\rho^{-1}\omega_a \nabla\eta$ (where ω_a = rot $\mathbf{u} + 2\omega$ is the absolute vorticity, with ω the angular velocity vector due to rotation of the planet), which have the form

(42) $$\left(\frac{d_h}{dt} + w\frac{\partial}{\partial z}\right)\eta = \varepsilon_1, \quad \left(\frac{d_h}{dt} + w\frac{\partial}{\partial z}\right)\rho^{-1}\omega_a \nabla\eta = \varepsilon_2,$$

where d_h/dt is the individual derivative with respect to the horizontal motion, and ϵ_1 and ϵ_2 are non-adiabatic factors (so that for adiabatic processes, when $\epsilon_1 = \epsilon_0 = 0$, the entropy and the potential vortex are Lagrangian invariants of the fluid particles; the first of these conditions is the original definition of an adiabatic process). We eliminate the quantity w from (42) and consider only very large-scale processes (for which the horizontal scales l of inhomogeneities in the thermohydrodynamic fields greatly exceed the thickness h of the layer), using the fact that such processes are quasi-horizontal, quasi-hydrostatic, and quasi-geostrophic (the last part means that in them the horizontal pressure gradient is approximately balanced out by the Coriolis force, and thus the deviation $p' = p - p_0$ of the pressure from the basic hydrostatic distribution $p_0(z)$ has the form $p' \approx \rho_0 f\psi$, where ψ is the horizontal flow function and $f = 2\omega_z$ is the Coriolis parameter). Then ([25], [47], [49], [58])

(43) $$\begin{cases} \dfrac{\partial \mathscr{L}\psi}{\partial t} + \dfrac{\partial(\psi, \mathscr{L}\psi)}{\partial(x, y)} + \beta\dfrac{\partial\psi}{\partial x} = \varepsilon_3, \\ \mathscr{L} = \Delta + \dfrac{\partial}{\partial z}\dfrac{h^2}{l_R^2}\dfrac{\partial}{\partial z}, \quad l_R = hNf^{-1}, \end{cases}$$

where x is directed along a circle of longitude, y is directed along a meridian, $\beta = \partial f/\partial y$, ϵ_3 describes the non-adiabatic factors, and l_R is called the radius of the Rossby deformation (N is the Väisälä-Brunt frequency mentioned above; in the Earth's atmosphere and ocean $N \gg f$, so that $l_R \gg h$; Nf^{-1} is the width of the spectrum of characteristic frequencies for the internal gravitational waves). The quantity $\omega_* = \mathcal{L}\psi + f$ can be called the potential vorticity, and $\Omega_* = (1/2)\overline{\omega_*^2}$ the potential enstrophy, and with this modification the statistical regularities listed in the preceding section for two-dimensional turbulence can be extended to very large-scale processes in the atmosphere and ocean (see [59] and [54]).

The principal difference between (43) and (32) is that the left-hand side of the former contains the term $\beta \partial \psi / \partial x$ describing the so-called Rossby-Blinova waves (whereas the second term in (32) and (43) describes non-linearly interacting vortices). The ratio δ of the second term to the third, that is, the "parameter of non-linearity" of the waves, is (as usual) equal to the ratio of the typical velocity U of motion of the fluid to the phase velocity c of the waves, which in the case of Rossby-Blinova waves with wave number k is equal to $c(k) = \beta(2k^2)^{-1}$. For $\delta < 1$ the waves dominate in the flow, for $\delta > 1$ the vortices dominate, and to the boundary value $\delta = 1$ there corresponds a horizontal scale $l = k^{-1}$ equal to

$$l_\beta = (2U/\beta)^{1/2}. \tag{44}$$

On the Earth for $\beta = 2.10^{-8}\ \mathrm{km}^{-1}\mathrm{s}^{-1}$ we have $l_\beta \sim 1000$ km in the atmosphere, where $U \sim 10$ m/s, while $l_\beta \sim 100$ km in the ocean, where $U \sim 10$ cm/s. If initially the field ψ represents a turbulence with scales $l < l_\beta$, then the mean scale l_1 with respect to the energies grows with time according to the law $\partial l_1/\partial t = aU$ mentioned in the previous section, and in a time of order l_β/aU attains the value l_β for which the vortices are transformed into Rossby-Blinova waves. Subsequently their mean scale continues to grow (but the frequency $\sigma = -\beta k_x k^{-2}$ will decrease) due to the resonance interactions between them (since in the resonance triad the wave with largest frequency turns out to be unstable with respect to an increase in energy in the other two waves), but more slowly than was the case for the vortices (since the wave interaction requires both the superposition of triples of waves in space and resonance between their wave numbers and frequencies). Because of the dispersion relation $\sigma = -\beta k_x k^{-2}$, the appearance of [illegible] tendency towards the motion of inhomogeneities of the field ψ westwards, while a decrease in σ leads to them becoming anisotropic—to a preferential growth of the meridian wave numbers k_y, that is, to a development of zonal flows. These effects were predicted by Rhines [60] and have been confirmed by his numerical experiments (which indicated that a is equal to 3.10^{-2} for vortices and to 6.10^{-3} for waves); they may be the mechanism for the formation of the bands on Jupiter (see [58]).

Another difference between (43) and (32) is the appearance of the second term in the operator $\mathcal{L}$. For $l < l_R$ the first term dominates in it, that is, the vertical interactions between the different layers of fluid play a small role, and these layers evolve approximately independently of one another, hence, the vertical inhomogeneities are not washed away (a "baroclinic" structure). Conversely, for $l > l_R$ the second term plays an essential role in $\mathcal{L}$, and different layers of fluid interact strongly along the vertical, so that the vertical inhomogeneities are washed away and the whole fluid evolves as a single layer (a "barotropic" structure). Thus, with the growth of $l_1(t)$ there arises a tendency towards "barotropization" of the vortices and waves (see [61] and [58]). In the atmosphere $l_R \sim 2000$ km, while in the ocean

$l_R \sim 50$ km. Hence, with increasing l_1 the evolution is chiefly "baroclinic vortices → baroclinic waves → barotropic waves" in the atmosphere, while it is chiefly "baroclinic vortices → barotropic vortices → barotropic waves" in the ocean (more precisely, $\delta \sim (l_\beta/l_R)^2 > 1$ for large l/l_R in the ocean, and, while very large barotropic vortices can be regarded as non-linear barotropic waves, linear waves are nevertheless not attained here).

References

[1] A.S. Monin and A.M. Yaglom, *Statisticheskaya gidromekhanika*, Parts 1, 2, Nauka, Moscow 1965, 1967.
Translation: Statistical fluid mechanics, Vols. 1, 2, MIT Press, Cambridge, MA 1971, 1975.

[2] L.W. Keller and A.A. Friedmann, Differentialgleichungen für die turbulente Bewegung einer kompressiblen Flüssigkeit, Proc. 1st Internat. Congress Appl. Mech., Delft 1924, 395-405. JBuch **51**-667.

[3] E. Hopf, Statistical hydromechanics and functional calculus, J. Rat. Mech. Anal. **1** (1952), 87-123.

[4] E.B. Gledzer and A.S. Monin, A method of diagrams in perturbation theory, Uspekhi Mat. Nauk **29**:3 (1974), 111-159. MR **53** #4815.
= Russian Math. Surveys **29**:3 (1974), 117-168.

[5] M.I. Vishik and A.V. Fursikov, *Matematicheskie zadachi statisticheskoi gidromekhaniki* (Mathematical problems of statistical hydromechanics), Nauka, Moscow 1980.

[6] A.S. Monin, The equations for the finite-dimensional probability distributions of a field of turbulence, Dokl. Akad. Nauk **177** (1967), 1036-1038. PA **68**-48031.
= Soviet Phys. Dokl. **12** (1968), 1113-1114.

[7] ———, The equations of turbulent motion, Prikl. Mat. Mekh. **31** (1967), 1057-1068. PA **68**-47923.
= Appl. Math. Mech. **31** (1967), 1057-1068.

[8] T.S. Lundgren, Distribution functions in the statistical theory of turbulence, Phys. Fluids **10** (1967), 969-975. PA **67**-31777.

[9] F.R. Ulinich and B.Ya. Lyubimov, The statistical theory of turbulence in an incompressible fluid for large Reynolds numbers, Zh. Eksper. Teoret. Fiz. **55** (1968), 951-965. PA **69**-3068.
= Soviet Physics JETP **28** (1968), 494-500.

[10] V.M. Ievlev, Approximate equations of turbulent incompressible fluid flow, Izv. Akad. Nauk SSSR Mekh. Zhidk. Gaza **1970**, no. 1, 91-103.
= Fluid Dynamics **5** (1970), 80-90.

[11] G.I. Barenblatt, *Podobie, avtomodel'nost', promezhutochnaya asimptotika,* Gidrometeoizdat, Leningrad 1978.
Translation: Similarity, self-similarity, intermediate asymptotics, Plenum Press, New York 1979.

[12] A.N. Kolmogorov, On degeneration of isotropic turbulence in an incompressible viscous fluid, C.R. (Doklady) Acad. Sci. URSS (N.S.) **31** (1941), 538-540. MR **3**-26.

[13] ———, The local structure of turbulence in an incompressible fluid for very large Reynolds numbers, C.R. (Doklady) Acad. Sci. URSS (N.S.) **30** (1941), 301-305. MR **2**-327.

[14] A.N. Kolmogorov, Dissipation of energy in locally isotropic turbulence, C.R. (Doklady) Acad. Sci. URSS (N.S.) **32** (1941), 16-18. MR **3**-221.

[15] A.S. Monin and A.M. Yaglom, On laws of very small-scale turbulent motions of liquids and gases, Uspekhi Mat. Nauk **18**:5 (1963), 93-114. MR **28** # 2748.
= Russian Math. Surveys **18**:5 (1963), 89-109.

[16] A.N. Kolmogorov, The equations of turbulent motion of an incompressible fluid, Izv. Akad. Nauk Ser. Fiz. **6** (1942), 56-58.

[17] A.C. Monin, Dynamic turbulence in the atmosphere, Izv. Akad. Nauk Ser. Geograf. Geofiz. **14** (1950), 232-254.

[18] J.C. Rotta, Statistische Theorie nichthomogener Turbulenz, Z. Phys. **129** (1951), 547-572; **131** (1951), 51-77. PA **51**-7590; **52**-3339.

[19] A.S. Monin, On symmetry properties of turbulence in the surface layer of air, Izv. Akad. Nauk SSSR Ser. Fiz. Atmosfer. i Okeana **1** (1965), 45-54. PA **65**-13245.
= Izv. Acad. Sci. USSR Atmospher. Ocean. Phys. **1** (1965), 25-30.

[20] ———, On the structure of the boundary layer of the atmosphere, Izv. Akad. Nauk SSSR Ser. Fiz. Atmosfer. i Okeana **1** (1965), 258-265.
= Izv. Acad. Sci. USSR Atmospher. Ocean. Phys. **1** (1965), 153-156.

[21] A.S. Monin, On a temperature-inhomogeneous boundary layer of the atmosphere, Izv. Akad. Nauk SSSR Ser. Fiz. Atmosfer. i Okeana **1** (1965), 490-500.
= Izv. Acad. Sci. USSR Atmospher. Ocean. Phys. **1** (1965), 287-292.

[22] ——— and L.M. Simuni, A model for the differential rotation of the Sun, Dokl. Akad. Nauk SSSR **250** (1980), 1348-1351. PA **81**-22769.
= Soviet Phys. Dokl. **25** (1980), 57-59.

[23] ———, *Solnechnyi tsikl* (The solar cycle), Gidrometeoizdat, Leningrad 1980.

[24] ——— and L.M. Simuni, Statistical hydromechanics of the solar differential rotation, Proc. Nat. Acad. Sci. USA **79** (1982), 3903-3907 (in English). PA **82**-106661.

[25] ——— and R.V. Ozmidov, *Okeanskaya turbulentnost'* (Ocean turbulence), Gidrometeoizdat, Leningrad 1981.

[26] A.M. Obukhov, Turbulence in a temperature [illegible] Teor. Geofiz. Akad. Nauk SSSR **1** (1946), 95-115.

[27] A.S. Monin, The turbulent mode in the surface layer of air, Inform. Sb. Gl. Upr. Gidrometeosluzhby **1950**, no. 1, 13-27.

[28] ——— and A.M. Obukhov, Dimensionless characteristics of turbulence in the surface layer of the atmosphere, Dokl. Akad. Nauk SSSR **93** (1953), 223-226.

[29] ——— and ———, Fundamental regularities of turbulent mixing in the surface layer of the atmosphere, Trudy Geofiz. Inst. Akad. Nauk SSSR **1954**, no. 24, 163-187.

[30] J.L. Lamley and H.A. Panofsky, The structure of atmospheric turbulence, Interscience Publ., New York 1964.

[31] A.B. Kazanskii and A.S. Monin, On the turbulent mode above the surface layer of air, Izv. Akad. Nauk SSSR Ser. Geofiz. **1960**, no. 1, 165-169.
= Bull. Acad. Sci. USSR Geophys. Ser. **1960**, no. 1, 110-112.

[32] ——— and ———, On the dynamic interaction between the atmosphere and the surface of the Earth, Izv. Akad. Nauk SSSR Ser. Geofiz. **1961**, no. 5, 786-788.
= Bull. Acad. Sci. USSR Geophys. Ser. **1961**, no. 5, 514-515.

[33] ——— and ———, On the turbulent mode in the surface layer of air under conditions of unstable stratification, Izv. Akad. Nauk SSSR Ser. Geofiz. **1958**, no. 6, 741-751.
= Bull. Acad. Sci. USSR Geophys. Ser. **1958**, no. 6, 421-426.

[34] A.M. Obukhov, On the structure of the temperature field and the velocity field under conditions of free convection, Izv. Akad. Nauk SSSR Ser. Geofiz. **1960**, no. 9, 1392-1396.
= Bull. Acad. Sci. USSR Geophys. Ser. **1960**, no. 9, 928-930.
[35] A.B. Kazanskii and A.S. Monin, Turbulence in surface inversions, Izv. Akad. Nauk SSSR Ser. Geofiz. **1956**, no. 1, 79-86.
[36] A.S. Monin, V.G. Neiman, and B.N. Filyushkin, On density stratification in the ocean, Dokl. Akad. Nauk SSSR **191** (1970), 1277-1279.
[37] A.M. Obukhov, On the distribution of energy in the spectrum of a turbulent flow, Dokl. Akad. Nauk SSSR **32** (1941), 22-24; Izv. Akad. Nauk SSSR Ser. Geofiz. **5** (1941), 453-466.
[38] ———, The structure of the temperature field in a turbulent flow, Izv. Akad. Nauk Ser. Geogr. Geofiz. **13** (1949), 58-69.
[39] S. Corrsin, On the spectrum of isotropic temperature fluctuations in an isotropic turbulence, J. Appl. Phys. **22** (1951), 469-473. Erratum: ibid. **22**, 1292. PA **51**-4170.
[40] A.M. Obukhov, On the influence of Archimedean forces on the structure of the temperature field in a turbulent flow, Dokl. Akad. Nauk SSSR **125** (1959), 1246-1248.
[41] R. Bolgiano, Jr., Turbulent spectra in a stably stratified atmosphere, J. Geophys. Res. **64** (1959), 2226-2229. PA **60**-12173.
[42] ———, Structure of turbulence in stratified media, J. Geophys. Res. **67** (1962), 3015-3023. PA **62**-24008.
[43] A.S. Monin, On the spectrum of turbulence in the temperature-inhomogeneous atmosphere, Izv. Akad. Nauk SSSR Ser. Geofiz. **1962**, no. 3, 397-407. PA **62**-21627.
= Bull. Acad. Sci. USSR Geophys. Ser. **1962**, no. 3, 266-271.
[44] ———, Turbulence and microstructure in the ocean, Uspekhi Fiz. Nauk **109** (1973), 333-354. PA **74**-64864.
= Soviet Phys.-Usp. **16** (1973), 121-131.
[45] G.I. Barenblatt, The dynamics of turbulent spots and intrusions in a stably stratified fluid, Izv. Akad. Nauk SSSR Ser. Fiz. Atmosfer. i Okeana **14** (1978), 195-206. PA 78-40558.
= Izv. Acad. Sci. USSR Atmospher. Ocean. Phys. **14** (1978), 139-145.
[46] ——— and A.S. Monin, On a possible mechanism for the phenomenon of discoidal formations in the atmosphere, Dokl. Akad. Nauk SSSR **246** (1979), 934-937.
[47] A.P. Mirabel' and A.S. Monin, Two-dimensional turbulence, Uspekhi Mekhaniki **2**:3 (1979), 47-95.
[48] V.M. Kamenkovich, M.N. Koshlyakov, and A.S. Monin, *Sinopticheskie vikhri v okeane* (Synoptic vortices in the ocean), Gidrometeoizdat, Leningrad 1982.
[49] A.S. Monin, *Vvedenie v teoriyu klimata* (Introduction to the theory of climate), Gidrometeoizdat, Leningrad 1982.
[50] V.I. Arnol'd, *Matematicheskie metody klassicheskoi mekhaniki,* Nauka, Moscow 1974. MR **57** # 14032.
Translation: Les méthodes mathématiques de la méchanique classique, Éditions Mir, Moscow 1976. MR **57** # 14033.
[51] E.A. Novikov, Spectral inequalities for two-dimensional turbulence, Izv. Akad. Nauk SSSR Ser. Fiz. Atmosfer. i Okeana **14** (1978), 668-671. PA **78**-69403.
= Izv. Acad. Sci. USSR Atmospher. Ocean. Phys. **14** (1978), 474-476.

[52] R.H. Kraichnan, Inertial ranges in two-dimensional turbulence, Phys. Fluids **10** (1967), 1417-1428. PA **68**-4442.
[53] G.K. Batchelor, Computation of the energy spectrum in homogeneous two-dimensional turbulence, Phys. Fluids, Suppl. 2, **12** (1969), 233-238. PA **70**-20299.
[54] B.L. Gavrilin, A.P. Mirabel', and A.S. Monin, On the energy spectrum of synoptic processes, Izv. Akad. Nauk SSSR Ser. Fiz. Atmosfer. i Okeana **8** (1972), 483-493. PA 72-48472.
= Izv. Acad. Sci. USSR Atmospher. Ocean. Phys. **8** (1972), 275-280.
[55] P.M. Saunders, Comment on "Wavenumber-frequency spectra of temperature in the free atmosphere", J. Atmospher. Sci. **29** (1972), 197-199. PA **72**-44687.
[56] C.E. Leith, Diffusion approximation for two-dimensional turbulence, Phys. Fluids **11** (1968), 671-673. PA **68**-21440.
[57] R.H. Kraichnan, Inertial ranges transfer in two- and three-dimensional turbulence, J. Fluid Mech. **47** (1971), 525-535. PA **71**-47778.
[58] A.P. Mirabel' and A.S. Monin, Geostrophic turbulence, Izv. Akad. Nauk Ser. Fiz. Atmosfer. i Okeana **16** (1980), 1011-1023. PA **81**-44201.
= Izv. Acad. Sci. USSR Atmospher. Ocean. Phys. **16** (1980), 749-757.
[59] J.G. Charney, Geostrophic turbulence, J. Atmospher. Sci. **28** (1971), 1087-1095.
[60] P.B. Rhines, Waves and turbulence on a beta-plane, J. Fluid Mech. **69** (1975), 417-443. PA 75-61485.
[61] ———, The dynamics of unsteady currents, in: The sea. Ideas and observations on progress in the study of seas, vol. 6, Wiley, Chichester 1977, 189-318. PA 78-16054.

Translated by H. McFaden

Institute of Oceanology
Academy of Sciences of the USSR

Received by the Editors 20 January 1983