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A Survey of Quadratic Systems

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1. INTRODUCTION

The general topological theory of two-dimensional autonomous systems

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

due to Poincaré and Bendixson, has assumed an almost definitive form. However, there remains the problem of obtaining more explicit information for special classes of such systems. To use a vague analogy, we have something corresponding to the general theory of functions of a complex variable. Can we construct something corresponding to the theory of elliptic functions? Since linear systems can be integrated by means of elementary functions, the simplest nontrivial case is where P and Q are relatively prime polynomials of degree at most 2, which are not both linear:

$$P(x, y) = \sum_{i+k=0}^{2} a_{ik}x^i y^k, \qquad Q(x, y) = \sum_{i+k=0}^{2} b_{ik}x^i y^k. \tag{2}$$

We shall call such systems *quadratic*.

A variety of significant physical problems lead to systems of this type. For example, the Emden-Fowler equation of astrophysics (Chandrasekhar [*11*])

$$(\xi^2\eta')' + \xi^\lambda \eta^n = 0$$

is transformed by the change of variables

$$x = \frac{\xi\eta'}{\eta}, \qquad y = \frac{\xi^{\lambda-1}\eta^n}{\eta'}, \qquad t = \ln|\xi|$$

into the system

$$\dot{x} = -x(1 + x + y)$$

$$\dot{y} = y(\lambda + 1 + nx + y).$$

Similarly, the Blasius equation of fluid mechanics (Weyl [41])

$$\eta''' + \eta\eta'' = 0$$

is transformed by the change of variables

$$x = \frac{\eta\eta'}{\eta''}, \qquad y = \frac{\eta'^2}{\eta\eta''}, \qquad t = \ln|\eta'|$$

into the system

$$\dot{x} = x(1 + x + y)$$
$$\dot{y} = y(2 + x - y).$$

The general system

$$\dot{x} = x(a_0 + a_1x + a_2y)$$
$$\dot{y} = y(b_0 + b_1x + b_2y)$$

occurs in mathematical biology in the theory of two competing species (Gause [17]) and systems of this form also arise in nonlinear mechanics (Mayer [29], [30], Butenin [9]). Quadratic systems arise in several problems of compressible fluid flow (Taylor [37], Jones [19], Kestin and Zaremba [22]). Other applications are given in Richardson [33], Guderley and Yoshihara [18], Coppel [13].

2. The Center Problem

Quadratic systems appear to have been discussed first by Buchel [8], but his work is mainly a collection of examples. The first general property of these systems to be investigated was the conditions for a center. In general, when the right sides of the system (1) are polynomials of arbitrary degree, the necessary and sufficient conditions for the existence of a center are expressed by the vanishing of infinitely many polynomials in their coefficients. By Hilbert's basis theorem these must all be consequences of a finite number of such relations. The problem of determining such a basis explicitly when the right sides are quadratic polynomials was first attacked by Dulac [15]. Unfortunately Dulac used a complex normal form and did not concern himself with questions of reality. However, the problem has now been completely solved through the subsequent work of Kapteyn [20], [21], Frommer [16], Bautin [1], Saharnikov [34], Belyustina [4], and Sibirskiĭ [35] ,[36]. The final result will be stated here, since it is quite simple and is not available in the western literature.

The equation

$$\frac{dy}{dx} = -\frac{x - ax^2 + (2b - \alpha)xy - cy^2}{y + bx^2 + (2c + \beta)xy + dy^2} \tag{3}$$

has a center at the origin if and only if one of the following three conditions is satisfied:

I. $a + c = b + d = 0$.

II. $\alpha(a + c) = \beta(b + d)$, $a\alpha^3 - (3b + \alpha)\alpha^2\beta + (3c + \beta)\alpha\beta^2 - d\beta^3 = 0$.

III. $\alpha + 5(b + d) = \beta + 5(a + c) = ac + bd + 2(a^2 + d^2) = 0$.

These conditions can be shown to be necessary by carrying out the first few steps of the classical procedures, due to Poincaré and Liapunov. That they are also sufficient follows from the fact that in each of the three cases the equation (3) can be integrated in terms of elementary functions. For example, in case III there is an integral of the form

$$\frac{[f(x, y)]^3}{[g(x, y)]^2} = \text{constant},$$

where f is a quadratic and g a cubic polynomial.

This property of integrability tells us much more than the answer to our original question. It also tells us how far out the closed curves surrounding the center extend, that a limit cycle and a center cannot coexist, and what other critical points the system can have. In fact, one can determine all possible phase portraits of a quadratic system when one critical point is a center (Frommer [*16*], Lukaševič [*27*], Latipov and Širov [*26*], Latipov [*25*]).

3. Cycles

Poincaré showed that if a system (1) has a weak focus (i.e., a critical point which is a center for the corresponding variational equation) then limit cycles can be made to appear in its neighborhood by varying slightly the right sides. The problem of determining the maximum number of limit cycles which can appear is related to the center problem. It was proved by Bautin [*1*], [*2*] that for quadratic systems this maximum number is three. This result was the starting point for the remarkable work of Petrovskiĭ and Landis [*31*], who attempted to show, by going over into the complex domain, that a quadratic system can have at most three limit cycles altogether. Professor Jurgen Moser tells me that in addition to the published errata [*32*] further mistakes have been found, which have still not been corrected.

Judgment on the validity of Petrovskiĭ and Landis' result must therefore be suspended.

However, several interesting properties of the limit cycles of quadratic systems have been established independently by Tung Chin-chu [*38*]. These results are given below as Theorems 1-4. (Theorem 1 was already stated without proof by Yeh Yen-chien [*42*].) Their proofs are based on the following simple

LEMMA. *Three critical points can never be collinear. On any straight line which is not composed of paths the total number of critical points and contacts is at most two. If there are two such points, R_1 and R_2, then the paths intersecting the segment ∞R_1 cross in the same sense as the paths intersecting $R_2\infty$ and in the opposite sense to the paths intersecting R_1R_2.*

Proof. If the straight line $ax + by + c = 0$ contained three critical points it would intersect each of the conic sections $P = 0$, $Q = 0$ in three points. Hence P and Q would have the common factor $ax + by + c$, contrary to the hypothesis that they are relatively prime. The isocline with the same slope as this straight line is the conic section $aP + bQ = 0$. If the line is not composed of paths, it meets this conic section in at most two points, which are its contacts and critical points. If there are two distinct points of intersection, R_1 and R_2, both segments ∞R_1 and $R_2\infty$ must belong to the domain $aP + bQ > 0$ and the segment R_1R_2 to the domain $aP + bQ < 0$, or vice versa.

THEOREM 1. *The interior of a closed path is a convex region.*

Assume on the contrary that there exist two points R_1, R_2 inside the closed path γ such that the segment R_1R_2 contains a point of γ. By displacing R_2 slightly we can suppose that the segment R_1R_2 contains a point exterior to γ and thus has at least two intersections with γ. Let S_1 be the first intersection of γ with the segment ∞R_1 and S_2 its last intersection with the segment $R_2\infty$. Then γ crosses the line R_1R_2 in opposite senses at S_1 and S_2. Therefore, by the lemma, the segment S_1S_2 contains exactly one contact or critical point, T say. Hence γ cannot intersect the line R_1R_2 on the open segment S_1T or on the open segment TS_2, which is a contradiction.

THEOREM 2. *There exists a unique critical point in the interior of each closed path.*

One knows from general theory that there is at least one critical point inside any closed path. Suppose that there existed two critical points R_1, R_2 inside the closed path γ. Let S_1 be the (first) intersection of γ with the segment ∞R_1 and S_2 its (last) intersection with the segment $R_2\infty$. Then γ

crosses the line R_1R_2 in opposite senses at S_1 and S_2, which contradicts the lemma.

It follows as a corollary that there cannot exist three closed paths situated as in Fig. 1.

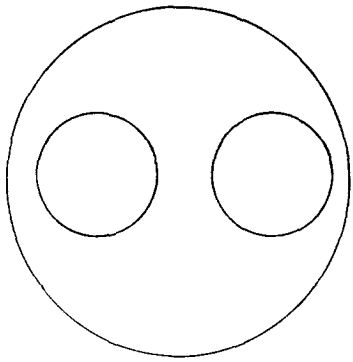


FIG. 1

THEOREM 3. *Two closed paths are oppositely oriented if their interiors have no common point.*

Let R_k be a critical point in the interior of the closed path γ_k $(k = 1, 2)$. If γ_1 and γ_2 were similarly oriented the segment R_1R_2 would be crossed in opposite senses at its intersections with γ_1 and γ_2. But this is impossible, because R_1R_2 is a segment without contact by the lemma.

It follows as a corollary that there cannot exist three closed paths situated as in Figure 2.

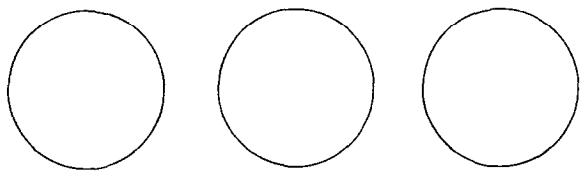


FIG. 2

THEOREM 4. *Two closed paths are similarly oriented if their interiors have a common point.*

If the interiors have a common point one closed path, γ_1, must be situated inside the other, γ_2. Let R be a critical point inside γ_1 and consider any straight line through R. If γ_1 and γ_2 were oppositely oriented, they would intersect the segment $R\infty$ in opposite senses, and so this segment would contain a contact or a critical point distinct from R. The same holds for the segment ∞R. But this contradicts the lemma.

Theorems 1-4 hold not only for closed paths but also for separatrix cycles, i.e., Jordan curves composed of paths and saddle-points (Tung Chin-chu [*39*]).

If Petrovskiĭ and Landis' theorem that there exist at most three limit cycles is valid, it follows that the only possible limit cycle configurations are those shown in Fig. 3. That there actually exist systems which have

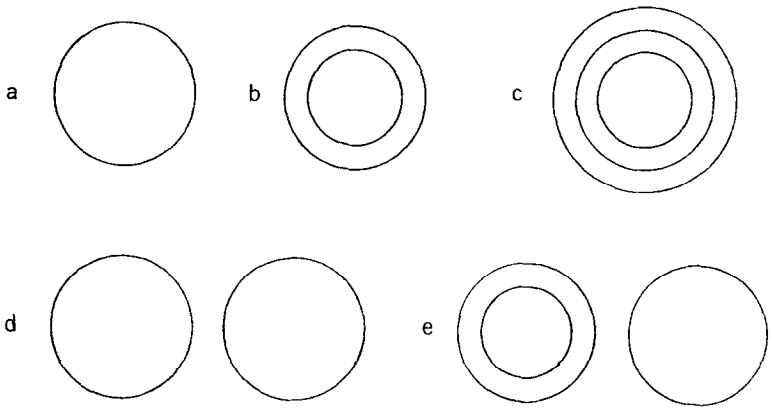


FIG. 3

limit cycles with these configurations was shown for (a) by Frommer [*16*], for (b) and (c) by Bautin [*2*], for (d) by Yeh Yen-chien [*42*] and Tung Chin-chu [*38*], and for (e) by Tung Chin-chu [*38*].

4. CRITICAL POINTS

It does not seem to have been observed previously that in a similar manner to Theorem 3 one can prove

THEOREM 5. *Two critical points, each of which is a focus or a center, are oppositely oriented.*

It follows as a corollary that the total number of foci and centers is at most two. This was first proved by Berlinskiĭ [*5*] and another proof has been given by Kukles and Casanova [*23*]. Both proofs are quite different from the present one and much less transparent.

It was demonstrated by Vorob'ev [*40*] that the critical point inside a closed path cannot be a node. However, his proof really shows that the critical point must be elementary and actually a focus or a center.

THEOREM 6. *A critical point in the interior of a closed path must be either a focus or a center.*

We suppose that the critical point is located at the origin and we introduce polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. By (2) the equation for θ has the form

$$\dot{\theta} = f_2(\theta) + rf_3(\theta),$$

where

$$f_2(\theta) = b_{10}\cos^2\theta + (b_{01} - a_{10})\cos\theta\sin\theta - a_{01}\sin^2\theta,$$

$$f_3(\theta) = b_{20}\cos^3\theta + (b_{11} - a_{20})\cos^2\theta\sin\theta + (b_{02} - a_{11})\cos\theta\sin^2\theta - a_{02}\sin^3\theta.$$

If the origin is neither a focus nor a center, then

$$(b_{01} - a_{10})^2 + 4a_{01}b_{10} \geqslant 0,$$

and so the equation $f_2(\theta) = 0$ has at least one real root. Since $f_3(\theta)$ is a homogeneous cubic in $\cos\theta$, $\sin\theta$ the equation $f_3(\theta) = 0$ also has at least one real root. If these two equations had a common root θ_1 the ray $\theta = \theta_1$ would be composed of critical points and paths, and the origin could not be surrounded by a closed path. It follows that there exists a sector $[\theta_1, \theta_2]$, with $0 < |\theta_1 - \theta_2| < 2\pi$, such that

$$f_2(\theta_1) = 0 = f_3(\theta_2)$$

and neither $f_2(\theta)$ nor $f_3(\theta)$ vanishes for any other value of θ in this sector. We may suppose that $-f_2(\theta)/f_3(\theta)$ is positive in the interior of the sector, since if it were negative we could consider instead the sector $[\theta_1 + \pi, \theta_2 + \pi]$. Then $\dot{\theta}$ is equal to $f_2(\theta_2)$ for $\theta = \theta_2$ and has the sign of $f_3(\theta_1)$, i.e., of $-f_2(\theta_2)$, for $\theta = \theta_1$. Therefore a path passing through a point of the sector $[\theta_1, \theta_2]$ will remain in it, either for all later or for all previous t. Thus again the origin cannot be surrounded by a closed path.

Since the sum of the indices of the critical points inside a closed path is equal to $+1$ Theorem 6 provides an alternative proof of Theorem 2.

Berlinskiĭ [5] has established a general property of quadratic systems which have the maximum number—four—of critical points. The following simple proof was given recently by Kukles and Casanova [23].

THEOREM 7. *Suppose that there are four critical points. If the quadrilateral with vertices at these points is convex then two opposite critical points are saddles and the other two are antisaddles (nodes, foci, or centers). But if the quadrilateral is not convex then either the three exterior vertices are saddles and the interior vertex an antisaddle or the exterior vertices are antisaddles and the interior vertex a saddle.*

By a preliminary linear transformation we may suppose that the critical points are located at the origin (0, 0), $A = (1, 0)$, $B = (0, 1)$, and $C = (\alpha, \beta)$, where $\alpha \neq 0$, $\beta \neq 0$ and $\alpha + \beta \neq 1$. Then the differential equations assume the form

$$\dot{x} = a_1x(x - 1) + b_1y(y - 1) + c_1xy,$$

$$\dot{y} = ax(x - 1) + by(y - 1) + cxy,$$

where

$$c = -\frac{\alpha - 1}{\beta} a - \frac{\beta - 1}{\alpha} b, \qquad c_1 = -\frac{\alpha - 1}{\beta} a_1 - \frac{\beta - 1}{\alpha} b_1 .$$

The Jacobian at the origin is $D_0 = a_1b - ab_1$. Since the right sides of the differential equations are not proportional, we have $D_0 \neq 0$. The origin is a saddle or antisaddle according as $D_0 \lessgtr 0$. It is easily verified that at the other critical points the Jacobian has the values

$$D_A = -\frac{\alpha + \beta - 1}{\alpha} D_0 ,$$

$$D_B = -\frac{\alpha + \beta - 1}{\beta} D_0 ,$$

$$D_C = (\alpha + \beta - 1) D_0 .$$

If $\alpha > 0$, $\beta > 0$, $\alpha + \beta > 1$ then the quadrilateral $OACB$ is convex, D_0 and D_C are of the same sign, D_A and D_B have the opposite sign. If $\alpha > 0$, $\beta > 0$, $\alpha + \beta < 1$, then the quadrilateral is not convex and D_C has the opposite sign to D_0, D_A, D_B. The other possible values of α, β may be discussed similarly.

5. Additional Properties

We have seen that systems of the form (1) with

$$P(x, y) = x(a_0 + a_1x + a_2y), \qquad Q(x, y) = y(b_0 + b_1x + b_2y),$$

occur frequently in applications. It has been shown by Bautin [3] that such a system cannot have a limit cycle. In fact, if there exists a closed path γ, it does not intersect the coordinate axes, since these are composed of paths. By changing the signs of x and y, if necessary, we can suppose that γ is situated in the first quadrant. Since there is a critical point in the interior of γ, the equations

$$a_0 + a_1x + a_2y = 0, \qquad b_0 + b_1x + b_2y = 0$$

have a solution. If $D = a_1b_2 - a_2b_1$ were equal to zero it would follow that also $a_0b_2 - a_2b_0 = 0$. Hence P and Q would have a common factor, which is contrary to hypothesis. Therefore $D \neq 0$.

Set

$$B(x, y) = x^{k-1}y^{h-1},$$

where

$$k = \frac{b_2(b_1 - a_1)}{D}, \qquad h = \frac{a_1(a_2 - b_2)}{D}.$$

It is easily verified that

$$\frac{\partial}{\partial x}(BP) + \frac{\partial}{\partial y}(BQ) = \frac{gB(x, y)}{D},$$

where

$$g = a_1b_0(a_2 - b_2) + a_0b_2(b_1 - a_1).$$

Now

$$0 = \int_\gamma B(\dot{x}\dot{y} - \dot{x}\dot{y})\,dt = \int_\gamma (BQdx - BPdy).$$

By Green's theorem the right side is equal to the double integral over the interior of γ of $-gB(x, y)/D$. Since $B > 0$ this leads to a contradiction, unless $g = 0$. But if $g = 0$ the system has a center and in fact has the integral

$$x^k y^h(a_1b_0x + a_0b_2y + a_0b_0) = \text{const.}$$

In conclusion we mention briefly some further results on quadratic systems. Berlinskiĭ [5], [6], [7] has determined all possible distributions of critical points. Latipov [24] has studied the critical points at infinity. Chin Yuan-shun [12] has shown that a quadratic system can have an ellipse as a limit cycle and has determined the possible phase portraits of such systems. Čerkas [10] has shown that a quadratic system can have an algebraic limit cycle of degree 3. Lyagina [28] ha sdetermined all possible phase portraits of *homogeneous* quadratic systems.

What remains to be done? Ideally one might hope to characterize the phase portraits of quadratic systems by means of algebraic inequalities on the coefficients. However, attempts in this direction have met with very limited success (Yeh Yen-chien [43], [44], Deng Yao-hua [14]). A more feasible problem seems to be that of determining all possible phase portraits of quadratic systems. Moreover, the solurion of this restricted problem would still be of considerable practical value.

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