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The Influence of Tidal Theory Upon the Development of Mathematics

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It is now 300 years since Torricelli made his famous experiment which led to the invention of the barometer and to the tests with it on a mountain at the suggestion of Descartes and Pascal. Just 200 years have elapsed since Clairaut published his work on the Figure of the Earth in which the general equations of hydrostatics were formulated and since d'Alembert published his Traité de Dynamique in which numerous applications were made of the principle which bears his name. As Daniel Bernoulli had already published his researches in hydrodynamics and Clairaut was just beginning his work on the motion of the moon under the gravitational attraction of the sun and the earth, the time was ripe for the creation of a dynamical theory of the tides in place of the equilibrium theory in which the rotation of the earth is treated as very slow.

In 1744 d'Alembert made a first attempt to derive hydrodynamical equations with the aid of his principle and two years later he made a second attempt when he attacked the very difficult problem of the influence on the atmosphere of the changing positions of the sun and moon. He did not solve this problem but obtained some interesting results as by-products. Some of the difficulties of the problem may, perhaps, be understood when it is realized that at this time mathematicians were interested chiefly in the theory of *undamped* oscillations of a mechanical system. This theory began with the work of Brook Taylor on the vibrating string. Jean Bernoulli became interested in this work and inspired his pupils Daniel Bernoulli and Leonard Euler to work along similar lines. A system capable of free vibration was supposed to be disturbed by a transient force and left in a state of free vibration. After a series of disturbances the residual oscillation naturally depends upon the past history. If the mechanical system has two natural periods of vibration which are nearly equal the residual oscillation may exhibit the phenomenon of beats, if the two periods are exactly equal the residual oscillation may be of increasing amplitude.* If the past life of the system is regarded as infinite the residual oscillation may be built up from an infinite series of terms and there may or may not be convergence.

In the case of a simple jolt represented by $f(t)=1$ for $0<t<a$ and $f(t)=0$ for $t<0$ and $t>a$, the solutions of the equations

$$d^2u/dt^2+m^2u=f(t)$$

$$d^4u/dt^4+(m^2+n^2)d^2u/dt^2+m^2n^2u=f(t)$$

$$d^4u/dt^4+2m^2d^2u/dt^2+m^4u=f(t)$$

are respectively $$m^2u=\cos m(t-a)-\cos mt \qquad t>a$$

$$u=\frac{\cos((nt)-\cos(mt)-\cos n(t-a)+\cos m(t-a)}{n^2(n^2-m^2)}-\frac{\cos(mt)-\cos m(t-a)}{n^2m^2} \qquad t>a$$

$$u=\frac{(t-a)\sin m(t-a)-t\sin mt}{2m^3}-\frac{\cos(mt)-\cos m(t-a)}{m^4} \qquad t>a$$

In the case of a sustained oscillation in which the force acts over a long period the general solution consists of a particular integral and a complementary function which depends upon the past history of the system. About the middle of the eighteenth century we might imagine the question being asked. What shall we do about the past? The answer given by Laplace[1] in 1775 was essentially "Forget about it." Mathematically speaking Laplace's recipe was a theory of damped vibrations in which the free vibrations were supposed to be rapidly damped out by a kind of viscous friction with the result that the remaining vibration would have the same period as the exciting force when this consists of a term of type $A\sin(nt+a)$. The assumption of such a form seems at first sight to require a knowledge of the future of the force as well as its past but it should be noticed that when the solution of the equation

$$d^2u/dt^2+2kdu/dt+(k^2+m^2)u=f(t)$$

*It was not realized until the time of Weierstrass (1858) that the case does not actually occur in a correct theory of small undamped oscillations of a dynamical system. The account of the theory given by Lagrange is thus in one point incorrect.

is expressed in the form

$$mu = \int_{-\infty}^{t} e^{-k(t-s)} \sin m(t-s) f(s) ds$$

appropriate to the case in which u and du/dt are zero for $t = -\infty$, we can, with little error, replace $-\infty$ by $t-T$ where T is a large integral multiple of the period of the force $f(s) = A \sin(n+a)$. Making now the substitution $s = t-T$ we get

$$mu = \int_{0}^{T} e^{-k(T-r)} \sin m(T-r) A \sin(nt+nr) dr$$

and this is of the form $B \sin(nt) + C \cos(nt)$ or $K \sin(nt+b)$. Thus a vibration having the same period as the force is obtained by taking into consideration only the past and not the future of the force. The theory of damped vibrations is of great importance in the design of many scientific instruments in which the reading is to give a more or less faithful representation of the force. It was used by Laplace to show that in some cases at least, the equilibrium theory gives a good representation of the tide.

Laplace was not content, however, with a theory based on equations involving only one variable. In 1776 he started to calculate the particular integral having the same period as the force when the earth's rotation is taken into consideration. He assumed that the ocean covered the whole earth and had a constant depth. For simplicity he neglected the internal friction and looked for solutions of his partial differential equation that depended only on the latitude. If the equilibrium height of the tide at a place is $H \sin^2\theta \cdot \cos 2\phi$ or $Hx^2 \cos 2\phi$ where θ is the colatitude and ϕ the hour angle of the disturbing body, Laplace replaced Hx^2 by an unknown quantity L and looked for cases in which L depends only on x, thus obtaining the differential equation

$$(1-x^2)x^2 d^2L/dx^2 - x dL/dx + 2(x^2 - 4 + 2ex^4)L = -9Hx^2$$

To obtain a solution having the proper form at the poles he put

$$L = K_2 x^2 + K_4 x^4 + \cdots$$

using first a method of approximation in which only a finite number of terms were used. Later in his Mécanique Céleste he used an infinite series and calculates the coefficients with the aid of the recurrence relation

$$2n(2n+6)K_{2n+4} - 2n(2n+3)K_{2n+2} + 4eK_{2n} = 0$$

Now K_2 is known and K_4 is at first sight arbitrary but it is necessary to make the series for L converge for $x=1$ and so Laplace chooses K_4 in such a way that the ratio K_{2n+2}/K_{2n} tends to zero as n tends to infinity. The expression for K_4/K_2 then takes the form of an fininite continued fraction the word infinite being used here in the same way as in the term infinite series.

Some writers like Airy and Ferrel[2] were doubtful about the validity of Laplace's procedure but as Lamb points out[3], many students read only the Mécanique Céleste which contains the finished product and ignored the original papers of Laplace in which more explanations are given. Lord Kelvin[4] became an enthusiastic advocate of Laplace's solution and showed where Airy was in error. Some misunderstanding has arisen by a slight confusion between the free vibrations having a period different from that of the force which Laplace eliminated by the introduction of damping and free vibrations having the period of the force which can only exist when the depth of the ocean has certain special values.

Laplace's method of using infinite continued fractions has been used for the solution of boundary problems associated with other linear differential equations such as Mathieu's equation and the equation defining the spheroidal wave functions. A slight modification of Laplace's method, ascribed by Forsyth[5] to Linstedt,[6] is used now to find the separation constants for the homogeneous equation and the associated periods of free vibration in many physical problems. Some uncertainty was felt about the convergence of the infinite continued fractions as may be judged, for instance, from the review of Linstedt's work in the Jahrbuch der Fortschritte der Mathematik. An attempt to discuss the convergence of the continued fractions used in the theory of the tides was made by K. Ogura[2] but, judging from the review in the Jahrbuch, his analysis requires modification. Lamb states[3] that the infinite continued fractions are convergent but does not give the test. A simple test which seems to meet the situation is provided by a modification of an old test which has been revived and made more definite by O. Szász[8] and W. Leighton.[9]

It is well known that if P_n/Q_n is the nth convergent of a continued fraction there are recurrence relations

$$P_n=b_nP_{n-1}+a_nP_{n-2},\quad Q_n=b_nQ_{n-2}+a_nQ_{n-2}$$

where a_n is the numerator and b_n the denominator at the nth stage of the continued fraction. Now if $P_n=b_nb_{n-1}\cdot\cdot b_1p_n,\quad Q=b_nb_{n-1}\cdot\cdot b_1q_n$ we have the equations $P_n/Q_n=p_n/q_n$ and

$$p_n=p_{n-1}+c_np_{n-2},\quad q_n=q_{n-1}+c_nq_{n-2}$$

consequently the contiued fraction is replaced by one in which the constituents b_n are all unity. For this type the test just mentioned states that it is sufficient for convergence that $|c_n| \leq \frac{1}{4}$ and that this bound for $|c_n|$ cannot be improved. The corresponding test for the convergence of the fraction in the original form is

$$|a_n/b_{n-1}b_n| \leq \tfrac{1}{4}$$

and this is the test that is applicable to the continued fractions which occur in tidal theory.

Laplace's theory has been developed by many writers such as Kelvin,[4] Rayleigh,[10] Darwin,[11] Lamb,[12], Hough[13] and Love[14]. Tables have been constructed by Doodson[15] and the theory has been applied to atmospheric oscillations by introducing an equivalent height for the atmosphere.[16] It is, perhaps, to the atmosphere that the theory is most directly applicable because the ocean does not cover the whole earth. The mathematical developments of tidal theory for the actual ocean consist partly of difficult analytical work for oceans with special boundaries such as meridians, parallels of latitude or ellipses and partly of simpler analysis applied to special regions in which there is much damping due to the water being shallow, the entrance to a basin being narrow, or to changes in level in the basin. Notable work has been done by French writers[17] who seem to like the methods of integral equations, by English writers like Proudman,[18] Doodson, Taylor, Jeffreys, Goldsbrough, Goldstein, Grace and Street who have done much numerical work in addition to elucidation of the theory and by writers of the Norwegian school whose work is summarized in the book of Bjerknes,[19] and his collaborators and in the paper of Solberg.[20] In the recent work of Proudman[21] a differential equation

$$d^2P/dx^2 + d^2P/dy^2 + e^{2x}(1 - w^2k^2)d^2P/dz^2 = 0$$

is obtained for the quantity $P = p/\rho + V$, where p denotes the pressure, ρ the density and V the potential of the gravitational and centrifugal force as in the work of Clairaut. The quantities e^x, y and z are the usual cylindrical co-ordinates, w is the angular speed of the earth and πk the period of oscillation. Solutions are given for some special regions.

In the review of the paper by A. Weinstein[22] it is pointed out that the equation is similar to one considered by M. Brillouin and J. Coulomb[23] in 1933 and that according to J. Hadamard[24] the boundary conditions that are generally used are suitable for equations of elliptic type only while the partial differential equation is of hyperbolic type when $w^2k^2 > 1$. This is a question in which the present author[25] has been much interested in connection with the theory of an elastic fluid.

Notable work on equations which change in type has been done by M. Cibrario.[26]

A comparison of tidal theory with observation may be obtained by analysing hourly records of sea level or atmospheric pressure in such a way that a constituent with a given period is separated from other variations that seem random in comparison. If, for instance, the 10 o'clock readings are averaged over a large number of days variations with periods different from 12 hours will on the average contribute little to the mean and the result will be a variation with change of the hour of reading that has the period of 12 hours. This may be supposed to represent the solar tide. The period of the lunar semi-diurnal tide exceeds that of the solar tide by about 1/28 of its value and so a different method of averaging is needed to separate out the lunar tide.

At the suggestion of Laplace observations over a long period were carried out at Brest, a place which was very suitable for such work as there was much friction to reduce accidental free oscillations of the water.

Readings of atmospheric pressure in the tropics show a well marked semidiurnal variation of pressure. This type of variation is also a constituent of the variation of pressure in higher latitudes but is generally masked by the pressure changes associated with the weather. Analysis of the observations indicated that the solar semidiurnal variation is large in comparison with the lunar semidiurnal variation, a result which was rather unexpected because the lunar tide generating force is more than twice as large as the solar tide generating force and even then is too small to account for the observed variation on an equilibrium theory. Indeed, a surface over which the moon's gravitational potential has an assigned constant value, moves up and down at a place during the day by about 1 metre and this corresponds on an equilibrium theory to a pressure variation of about 1/40 of a millimetre.

To account for the magnitude of the semidiurnal variation of pressure Laplace suggested that it might be of thermal origin but Lord Kelvin[27] raised the objection that then one would expect there to be a well marked diurnal variation over the whole earth which is contrary to experience. As a matter of fact there are local diurnal variations of pressure as indicated by the land and sea breezes of coastal regions and the mountain and valley regional breezes of the Alps which have been ably discussed by E. Ekhart.[28] There is also a pressure variation which is less local and has a 8-hour period. This has a well marked seasonal change and is thought by Bartels[29] to be of thermal origin because a harmonic analysis of the intensity of solar radiation indicates that the term with a 8-hour period has a seasonal change of just this type and a geographical distribution which, like that of the 6-hour pressure variation, can be represented roughly by means of a spherical harmonic of type p_4^3.

The existence of a 8-hour pressure variation was noted after J. Hann had made a long and careful study of the semidiurnal variation. According to Simpson,[30] this latter variation may be represented with some degree of accuracy by

$$p_2 = 0.937 \sin^3\theta \cdot \sin(2t+154°) + 0.137(\cos^2\theta - \tfrac{1}{3})\sin(2t-2\phi+105°)$$

the unit being lmm. of mercury. The first term is regarded by Chapmen[31] to be the result of both the sun's tidal and thermal action which are about equally effective. He was able to account for the phase of this wave on a resonance theory in which the magnification is about a hundred fold. The resonance theory was put forward by Lord Kelvin[27] in 1882 and was developed by M. Margules[32] who considered the oscillations of the atmosphere of a spheroidal earth. Chapman,[33] Pramanik and Topping have discussed the solar and lunar barometric pressure oscillations in the light of the resonance theory and have found that on this theory four types of pressure oscillations should be magnified by resonance. The theory of resonance is not so simple when the amplitude of vibration varies with position as it does in the atmosphere. A well known case which may be used for purposes of illustration is the case of a circular membrane. It was thought by Savart that a membrane would respond to almost any frequency above a certain limit but Bourget[34] found by a study of the roots of equations of type $J_n(x)=0$ that the possible frequencies seem to form an enumerable set. He surmised indeed that if m and n are positive integers the positive roots of $J_m(x)=0$ differ from those of $J_n(x)=0$ and from those of $J_0(x)=0$, there being no common root of any of these equations. This has not yet been proved but it is known to be true for the first few values of m and n. The nodal circles and radii of the different type of normal modes of vibration seem also to be different in each case. When the membrane is in a state of forced vibration the distribution of energy amongst the different normal modes depends very much upon the distribution over the membrane of the exciting forces. In the case of a single force acting at a point of the membrane a particular mode will have no energy at all if one of its nodal lines passes through the point. There is in fact a kind of geometrical resonance in addition to that which depends upon a concordance of periods. This may be illustrated, perhaps, by taking the case of a vibration problem in which the forced vibration having the same period as the force is given by the solution of an integral equation

$$f(s)=F(s)-k\int_0^1 g(s,t)F(t)dt$$

in which k depends on the period of the force and $f(s)$ upon its local distribution. To find the function $F(s)$ which gives the local variation of displacement it is advantageous to expand $f(s)$ in a series of the functions $\psi_n(s)$ which satisfy the homogeneous equation

$$\psi_n(s)=k_n\int_0^1 g(s,t)\psi_n(t)dt$$

and correspond to the free vibrations of the system. If

$$f(s)=\sum c_n\psi_n(s)$$

and

$$F(s)=\sum C_n\psi_n(s)$$

we have the equation

$$\sum c_n\psi_n(s)\sum C_n\psi_n(s)-k\int_0^1 g(s,t)\sum C_n\psi_n(t)dt$$

$$=\sum C_n\psi_n(s)-\sum (k/k_n)C_n\psi_n(s).$$

Hence

$$c_n=C_n(1-k/k_n) \quad \text{or} \quad C_nk_n/(k_n-k).$$

The value of C_n may be large because k is nearly equal to k_n or it may be large because c_n is large, there are thus two kinds of resonance. In order that a force may excite a particular mode of free vibration or produce a vibration something like it the force must get a toe hold so to speak at a place where the vibration to be imitated has a concentration of energy.

This point is illustrated in the theory of atmospheric oscillations developed recently by C. L. Pekeris.[35] A distribution of temperature with height is found such that the atmosphere has a 12-hour oscillation such as is required for the resonance theory of pressure oscillations and also a $10\frac{1}{2}$ hour oscillation of the type required to account by the theory of G. I. Taylor[36] for the velocity of propagation of the long waves produced by the eruption of Krakatoa. In the latter oscillation the energy is concentrated in the lower part of the atmosphere and there is a level at which the vertical velocity of the air is zero. In the 12 hour oscillation the amplitude and density of energy increase with height and there is a change of phase in the pressure oscillation such as is required by Chapman's dynamo theory of the diurnal variation of the earth's magnetic force.[37] On account of the large damping by viscous forces at a level of 200 kilometers this type of oscillation requires a periodic force of permanent type for its maintenance and inequalities in it arising from accidental disturbances such as the eruption of Krakatoa may be expected to be rapidly eliminated.

Pekeris shows that the existence of a semidiurnal free oscillation in accordance with the resonance theory restricts in some measure the possible distributions of temperature in the upper atmosphere. The distribution adopted by Pekeris is one in which there is a constant lapse rate up to a height of about 10 km. then an isothermal region with an absolute temperature of about 240 up to a height of about 38 km. This in turn is followed by a rise in temperature with a gradient about equal in magnitude to the former lapse rate until a maximum temperature of about 350 K is attained at a height of about 62 km. A second lapse rate of the same magnitude brings the temperature back to its former constant value at a height of about 79 km. and the temperature subsequently remains at this value. The two gradients in the region 38-79 km. are slightly greater in magnitude than that near the ground. Pekeris has also made computations for some slightly different distributions of temperature.

In 1910 H. Lamb[38] discussed the oscillations (mainly horizontal) of an atmosphere with an assumed temperature distribution for the case of an earth without rotation. His differential equation for the divergence of velocity is of the second order and involves both the velocity of sound for the assumed distribution of temperature and also the quantities σ and k which occur in the factor $e^{i\sigma t}J_0(kr)$ which gives the dependence on time and radius in the horizontal plane, the independent variable in the differential equation being the height above sea level. Lamb considers in particular the case of an atmosphere with uniform lapse rate and obtains a transcendental equation involving the confluent hypergeometric function which is hard to solve without suitable tables. In the case when σ^2/gk is very small, g being the acceleration of gravity, the equation is replaced by

$$\tfrac{1}{2}wJ_{n+1}(w) = \left(\frac{\beta}{\beta} - 1\right) J_n(w)$$

where $J_n(w)$ is the Bessel function of order n, β gives the lapse rate of the actual temperature distribution and β_1 gives the particular lapse rate in which the atmosphere is in convective equilibrium. Now this equation has been much studied as it occurs in the theory of vibration of circular plates, in the theory of the conduction of heat and in some electrical problems. A particular case of the equation was, I think, first obtained by Poisson.[39] The equation occurs in the work of Moore[40] and Hobson[41] on expansions in series of Bessel functions. The equation has an infinite number of roots which are all real when the constants satisfy certain conditions. Ways of calculating the roots have been given by McMahon, Airey and others. For the large roots the following method may be used. Writing the equation in the form

$$wJ_{n+1}(w) = xJ_n(w)$$

where n is a known constant and x is a parameter whose value can be changed, we find from the recurrence relations for the Bessel functions that w satisfies the differential equation

$$(w^2 - 2nx + x^2)dw/dx = w.$$

We try to satisfy this by means of a series of type

$$w^2 = W^2 + C + C_1 W^{-2} + C_2 W^{-4} + C_3 W^{-6} + \cdots$$

where $$W = \tfrac{1}{4}\pi(2n+1+4s),$$

s is a large integer and C, C_1, $C_2, \cdots$ functions of x to be determined. In trying to find these we shall use the property that when $x=0$ and $2n=-1$ or -3 the quantity W is always a root of the equation and so C, C_1, $C_2, \cdots$ should in this case be zero. A further fact which will be useful is that when $x=2n$ the equation reduces to $J_{n-1}(w)=0$ and so the formula should in this case agree with that obtained by putting $x=0$ replacing $n+1$ by $n-1$ and s by $s+1$. Using primes to denote differentiations with regard to x we find that we have to satisfy the equation

$$(W^2 + C - 2nx + x^2 + C_1 W^{-2} + C_2 W^{-4} + \cdots)(C' + C_1' W^{-2} + C_2' W^{-4} + \cdots)$$
$$= 2W^2 + 2C + 2C_1 W^{-2} + 2C_2 W^{-4} + \cdots$$

Equating coefficients we find that $C'=2$, $C_1' = 4nx - 2x^2$, etc. The first equations may be satisfied by writing

$$C = 2x - k(n+\tfrac{1}{2})(n+3/2)$$

To find the value of the constant k we put $x=2n$ and remark that in this case C should be equal to $-k(n-\tfrac{1}{2})(n-3/2)$, consequently $k=1$. For convenience we now write $C=2x-m$ where $m=(n+\tfrac{1}{2})(n+3/2)$. The differential equation for C_1 may be satisfied by writing

$$C_1 = 2nx^2 - (2/3)x^3 + am + bm^2$$

and when $x=2n$ we should have $C_1 = aM + bM^2$ where $M=(n-\tfrac{1}{2})(n-3/2)$. Hence we find that $a=\tfrac{1}{2}$, $b=-\tfrac{1}{3}$. This process may be continued, the constant of integration in C_2 being taken to be

$Am+Hm^2+Km^3$ and when $x=2n$ the value of C_2 should be $AM+HM^2+KM^3$. The series found by McMahon for the case $x=0$ can, when squared, be used as a check. Another check may be obtained by putting $n=-\frac{1}{2}$ and keeping x different from zero. The series for w can then be found by means of Lagrange's expansion using a method used successfully by Cauchy. Lamb's theory of atmospheric oscillations on a non-rotating earth has been used effectively by G. I. Taylor[42] who compares Lamb's differential equation with one obtained for the free oscillation of an atmosphere having the same temperature distribution upwards and enveloping the rotating earth. Taylor finds in fact that the limiting value of the velocity of propagation of waves of long period determines the periods of oscillation. The propagation of a pulse in the atmosphere has been studied by G. I. Taylor,[42] F. J. W. Whipple[43] and C. L. Pekeris,[44] the velocity of propagation is not greatly different from the ordinary velocity of propagation of sound.

In closing this discussion of the influence of tidal theory on the development of mathematics, reference should be made to the great advances in harmonic analysis and synthesis which have been made for the purpose of working up the observations and analyzing the tide producing forces. Many machines have been invented and a tide predicting machine was constructed by Lord Kelvin in 1876. Tests have been made to determine the accuracy and improvements in machines and methods of analysis have been made in recent years.[45] The literature of the subject is now enormous and so it cannot be summarized here.

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Abbreviations.

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