Applied Mathematics Is Bad Mathematics

Paul R. Halmos

It isn't really (applied mathematics, that is, isn't really bad mathematics), but it's different.

Does that sound as if I had set out to capture your attention, and, having succeeded, decided forthwith to back down and become conciliatory? Nothing of the sort! The "conciliatory" sentence is controversial, believe it or not; lots of people argue, vehemently, that it (meaning applied mathematics) is not different at all, it's all the same as pure mathematics, and anybody who says otherwise is probably a reactionary establishmentarian and certainly wrong.

If you're not a professional mathematician, you may be astonished to learn that (according to some people) there are different kinds of mathematics, and that there is anything in the subject for anyone to get excited about. There are; and there is; and what follows is a fragment of what might be called the pertinent sociology of mathematics: what's the difference between pure and applied, how do mathematicians feel about the rift, and what's likely to happen to it in the centuries to come?

What is it?

There is never any doubt about what mathematics encompasses and what it does not, but it is not easy to find words that describe precisely what it is. In many discussions, moreover, mathematics is not described as a whole

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but is divided into two parts, and not just in one way; there are two kinds of mathematics according to each of several different systems of classification.

Some of the dichotomies are well known, and others less so. Mathematics studies sizes and shapes, or, in other words, numbers (arithmetic) and figures (geometry); it can be discrete or continuous; it is sometimes finite and sometimes infinite; and, most acrimoniously, some of it is pure (useless?) and some applied (practical?). Different as these classification schemes might be, they are not unrelated. They are, however, not of equal strengths; the size-shape division, for instance, is much less clear-cut, and much less divisive, than the pure-applied one.

Nobody is forced to decide between vanilla ice cream and chocolate once and for all, and it is even possible to mix the two, but most people usually ask for the same one. A similar (congenital?) division of taste exists for mathematicians. Nobody has to decide once and for all to like only algebra (discrete) or only topology (continuous), and there are even flourishing subjects called algebraic topology and topological algebra, but most mathematicians do in fact lean strongly toward either the discrete or the continuous.

Squares and spheres

It would be a shame to go on and on about mathematics and its parts without looking at a few good concrete examples, but genuine examples are much too technical to describe in the present context. Here are a couple of artificial ones (with some shortcomings, which I shall explain presently).

Suppose you want to pave the floor of a room whose shape is a perfect square with tiles that are themselves squares so that no two tiles are exactly the same size. Can it be done? In other words, can one cover a square with a finite number of non-overlapping smaller squares all of which have different side-lengths? This is not an easy question to answer.

Here is another puzzle: if you have a perfect sphere, like a basketball, what's the smallest number of points you can mark on it so that every point on the surface is within an inch of one of the marked ones? In other words, what's the most economical way to distribute television relay stations on the surface of the globe?

Is the square example about sizes (numbers) or shapes (figures)? The answer seems to be that it's about both, and so is the sphere example. In this respect the examples give a fair picture; mixed types are more likely to occur (and are always more interesting) than the ones at either extreme. The examples have different flavors, however. The square one is more nearly arithmetic, discrete, finite, pure, and the one about spheres leans toward being geometric, continuous, infinite, applied.

The square problem is of some mild interest, and it has received attention in the professional literature several times, but it doesn't really have the respect of most mathematicians. The reason is not that it is obviously useless in the practical sense of the word, but that it is much too special (petty?, trivial?), in the sense of being isolated from most of the rest of mathematics and requiring *ad hoc* methods for its solution. It is not really a fair example of pure mathematics.

The sphere example, on the other hand, is of a great deal of practical use, but, nevertheless, it is not a fair example of applied mathematics: it is much easier (and much purer) than most applied problems, and, in particular, it does not involve motion, which plays the central role in the classical conception of what applied mathematics is all about.

Still, for what they are worth, here they are, and it might help to keep them in mind as the discussion proceeds.

Fiction or action

The pure and applied distinction is visible in the arts and in the humanities almost as clearly as in the sciences: witness Mozart versus military marches, Rubens versus medical illustrations, or Virgil's *Aeneid* versus Cicero's *Philippics*. Pure literature deals with abstractions such as love and war, and it tells about imaginary examples of them in emotionally stirring language. Pure mathematics deals with abstractions such as the multiplication of numbers and the congruence of triangles, and it reasons about Platonically idealized examples of them with intellectually convincing logic.

There is, to be sure, one sense of the word in which all literature is "applied". Shakespeare's sonnets have to do with the everyday world, and so does Tolstoy's *War and Peace*, and so do Caesar's commentaries on the wars he fought; they all start from what human beings see and hear, and all speak of how human beings move and feel. In that same somewhat shallow sense all mathematics is applied. It all starts from sizes and shapes (whose study leads ultimately to algebra and geometry), and it reasons about how sizes and shapes change and interact (and such reasoning leads ultimately to the part of the subject that the professionals call analysis).

There can be no doubt that the fountainhead, the inspiration, of all literature is the physical and social universe we live in, and the same is true about mathematics. There is also no doubt that the physical and social universe daily affects each musician, and painter, and writer, and mathematician, and that therefore a part at least of the raw material of the artist is the world of facts and motions, sights and sounds. Continual contact between the world and art is bound to change the latter, and perhaps even to improve it.

The ultimate goal of "applied literature", and of applied mathematics, is

action. A campaign speech is made so as to cause you to pull the third lever on a voting machine rather than the fourth. An aerodynamic equation is solved so as to cause a plane wing to lift its load fast enough to avoid complaints from the home owners near the airport. These examples are crude and obvious; there are subtler ones. If the biography of a candidate, a factually correct and honest biography, does not directly mention the forthcoming election, is it then pure literature? If a discussion of how mathematically idealized air flows around moving figures of various shapes, a logically rigorous and correct discussion, does not mention airplanes or airports, is it then pure mathematics? And what about the in-between cases: the biography that, without telling lies, is heavily prejudiced, and the treatise on aerodynamics that, without being demonstrably incorrect, uses cost-cutting rough approximations—are they pure or applied?

Continuous spectrum

Where are the dividing lines in the chain from biography to interpretive history to legend to fiction? We might be able to tell which of Toynbee, Thucydides, Homer, and Joyce is pure and which is applied, but if we insert a dozen names between each pair of them, as we pass from interpreted fact to pure fancy, the distinctions become blurred and perhaps impossible to define. The mathematical analogy is close: if we set out to sort a collection of articles that range from naval architecture to fluid dynamics to partial differential equations to topological vector spaces, the pure versus applied decisions that are clear at the two ends of the spectrum become fuzzy in the middle.

Pure mathematics can be practically useful and applied mathematics can be artistically elegant.

To confuse the issue still more, pure mathematics can be practically useful and applied mathematics can be artistically elegant. Pure mathematicians, trying to understand involved logical and geometrical interrelations, discovered the theory of convex sets and the algebraic and topological study of various classes of functions. Almost as if by luck, convexity has become the main tool in linear programming (an indispensable part of modern economic and industrial practice), and functional analysis has become the main tool in quantum theory and particle physics. The physicist regards the applicability of von Neumann algebras (a part of functional analysis) to elementary particles as the only justification of the former; the mathematician regards the connection as the only interesting aspect of the latter. *De gustibus non disputandum est*?

Just as pure mathematics can be useful, applied mathematics can be

more beautifully useless than is sometimes recognized. Applied mathematics is not engineering; the applied mathematician does not design airplanes or atomic bombs. Applied mathematics is an intellectual discipline, not a part of industrial technology. The ultimate goal of applied mathematics is action, to be sure, but, before that, applied mathematics is a part of theoretical science concerned with the general principles behind what makes planes fly and bombs explode.

The differences between people are sometimes as hard to discern as the differences between subjects, and it can even happen that one and the same person is both a pure and an applied mathematician. Some applied mathematicians (especially the better ones) have a sound training in pure mathematics, and some pure mathematicians (especially the better ones) have a sound training in applicable techniques. When the occasion for a crossover arises (a pure mathematician successfully solves a special case of the travelling salesman problem that arises in operations research, a relativity theorist brilliantly derives a formula in 4-dimensional differential geometry), each one is secretly more than a little proud: "See! I can do that stuff too!"

Doers and knowers

What I have said so far is that in some sense all mathematics is applied mathematics and that on some level it is not easy to tell the pure from the applied. Now I'll talk about the other side: pure and applied are different indeed, and if you know what to look for, and have the courage of your convictions, you can always tell which is which. My purpose is to describe something much more than to prove something. My hope is neither to convert the pagan nor to convince the agnostic, but just to inform the traveller from another land: there are two sects here, and these are the things that they say about each other.

The difference of opinion is unlike the one in which one sect says "left" and the other says "right"; here one sect says "we are all one sect" and the other says "oh no, we're not, we are two". That kind of difference makes it hard to present the facts in an impartial manner; the mere recognition that a conflict exists amounts already to taking sides. There is no help for that, so I proceed, with my own conclusions admittedly firm, to do the best I can for the stranger in our midst.

Human beings want to know and to do. People want to know what their forefathers did and said, they want to know about animals and vegetables and minerals, and they want to know about concepts and numbers and sights and sounds. People want to grow food and to sew clothes, they want to build houses and to design machines, and they want to cure diseases and to speak languages.

The doers and the knowers frequently differ in motivation, attitude,

technique, and satisfaction, and these differences are visible in the special case of applied mathematicians (doers) and pure mathematicians (knowers). The motivation of the applied mathematician is to understand the world and perhaps to change it; the requisite attitude (or, in any event, a customary one) is one of sharp focus (keep your eye on the problem); the techniques are chosen for and judged by their effectiveness (the end is what's important); and the satisfaction comes from the way the answer checks against reality and can be used to make predictions. The motivation of the pure mathematician is frequently just curiosity; the attitude is more that of a wide-angle lens than a telescopic one (is there a more interesting and perhaps deeper question nearby?); the choice of technique is dictated at least in part by its harmony with the context (half the fun is getting there); and the satisfaction comes from the way the answer illuminates unsuspected connections between ideas that had once seemed to be far apart.

The challenge [for the pure mathematician] is the breathtakingly complicated logical structure of the universe, and victory is permanent.

The last point deserves emphasis, especially if you belong to the large group of people who proudly dislike mathematics and regard it as inglorious drudgery. For the pure mathematician, his subject is an inexhaustible source of artistic pleasure: not only the excitement of the puzzle and the satisfaction of the victory (if it ever comes!), but mostly the joy of contemplation. The challenge doesn't come from our opponent who can win only if we lose, and victory doesn't disappear as soon as it's achieved (as in tennis, say); the challenge is the breathtakingly complicated logical structure of the universe, and victory is permanent (more like recovering precious metal from a sunken ship).

The basic differences in motivation, attitude, technique, and satisfaction are probably connected with more superficial but more noticeable differences in exposition. Pure and applied mathematicians have different traditions about clarity, elegance, and perhaps even logical rigor, and such differences frequently make for unhappy communication.

The hows and the whys listed above are not offered as a checklist to be used in distinguishing applied science from pure thought; that is usually done by a sort of intuitive absolute pitch. The word "spectrum" gives an analogical hint to the truth. In some sense red and orange are the same—just waves whose lengths differ a bit—and it is impossible to put your finger on the spot in the spectrum where red ends and orange begins—but, after that is granted, red and orange are still different, and the task of telling them apart is almost never a difficult one.

Beauty and boredom

Many pure mathematicians regard their specialty as an art, and one of their terms of highest praise for another's work is "beautiful". Applied mathematicians seem sometimes to regard their subject as a systematization of methods; a suitable word of praise for a piece of work might be "ingenious" or "powerful".

Mathematics, . . . despite its many subdivisions and their enormous rates of growth, . . . is an amazingly unified intellectual structure.

Here is another thing that has frequently struck me: mathematics (pure mathematics), despite its many subdivisions and their enormous rate of growth (started millennia ago and greater today than ever before), is an amazingly unified intellectual structure. The mathematics that is alive and vigorous today has so many parts, and each is so extensive, that no one can possibly know them all. As a result, we, all of us, often attend colloquium lectures on subjects about which we know much less than an average historian, say, knows about linguistics. It doesn't matter, however, whether the talk is about unbounded operators, commutative groups, or parallelizable surfaces; the interplay between widely separated parts of mathematics always shows up. The concepts and methods of each one illuminate all others, and the unity of the structure as a whole is there to be marvelled at.

That unity, that common aesthetic insight, is mostly missing between pure and applied mathematics. When I try to listen to a lecture about fluid mechanics, I soon start wondering and puzzling at the (to me) *ad hoc* seeming approach; then the puzzlement is replaced by bewilderment, boredom, confusion, acute discomfort, and, before the end, complete chaos. Applied mathematicians listening to a lecture on algebraic geometry over fields of non-zero characteristic go through a very similar sequence of emotions, and they describe them by words such as inbred, artificial, baroque folderol, and unnecessary hairsplitting.

It might be argued that from the proper, Olympian, impartial scientific point of view both sides are wrong, but perhaps to a large extent both are right—which would go to prove that we are indeed looking at two subjects, not one. To many pure mathematicians applied mathematics is nothing but a bag of tricks, with no merit except that they work, and to many applied mathematicians much of pure mathematics deserves to be described as meaningless abstraction for its own sake with no merit at all. (I mention in passing that in moments of indulgent self-depreciation the students of one particular branch of pure mathematics, category theory, refer to their branch as "abstract nonsense"; applied mathematicians tend to refer to it the same way, and they seem to mean it.)

New heresy

Some say that the alleged schism between pure and applied mathematics is a recent heresy at which the founding greats would throw up their hands in horror—the world is going to the dogs! There is a pertinent quotation from Plato's *Philebus*, which doesn't quite refute that statement, but it's enough to make you think about it again.

"Socrates: Are there not two kinds of arithmetic, that of the people and that of the philosophers? . . . And how about the arts of reckoning and measuring as they are used in building and in trade when compared with philosophical geometry and elaborate computations—shall we speak of each of these as one or two?

"Protarchus: . . . I should say that each of them was two."

Is the distinction that Socrates is driving at exactly the pure-applied one? If not that, then what?

The only other curiosity along these lines that I'll mention is that you can usually (but not always) tell an applied mathematician from a pure one just by observing the temperature of his attitude toward the same-different debate. If he feels strongly and maintains that pure and applied are and must be the same, that they are both mathematics and the distinction is meaningless, then he is probably an applied mathematician. About this particular subject most pure mathematicians feel less heat and speak less polemically: they don't really think pure and applied are the same, but they don't care all that much. I think what I have just described is a fact, but I confess I can't help wondering why it's so.

New life

The deepest assertion about the relation between pure and applied mathematics that needs examination is that it is symbiotic, in the sense that neither can survive without the other. Not only, as is universally admitted, does the applied need the pure, but, in order to keep from becoming inbred, sterile, meaningless, and dead, the pure needs the revitalization and the contact with reality that only the applied can provide.

The first step in the proof of the symbiosis is historical: all of pure mathematics, it is said, comes from the real world, the way geometry, according to legend, comes from measuring the effect of the floods of the Nile. (If that's false, if geometry existed before it was needed, the symbiosis argument begins on a shaky foundation. If it's true, the argument tends to prove only that applied mathematics cannot get along without pure, as an anteater cannot get along without ants, but not necessarily the reverse.)

Insofar as all mathematics comes from the study of sizes (of *things*) and shapes (of *things*), it is true that all mathematics comes from the things of the real world. Whether renewed contact with physics or psychology or

biology or economics was needed to give birth to some of the greatest parts of 20th century mathematics (such as Cantor's continuum problem, the Riemann hypothesis, and the Poincaré conjecture) is dubious.

The crux of the matter is, however, not historical but substantive. By way of a parable, consider chess. Mathematicians usually but sometimes grudgingly admit that chess is a part of mathematics. They do so grudgingly because they don't consider chess to be "good" mathematics; from the mathematical point of view it is "trivial". No matter: mathematics it is, and pure mathematics at that.

Applied mathematics cannot get along without pure, as an anteater cannot get along without ants, but not necessarily the reverse.

Chess has not been conceptually revitalized in many hundreds of years, but is vigorously alive nevertheless. There are millions of members of chess clubs, and every now and then the whole civilized world spends days watching Bobby Fischer play Boris Spassky. Chess fires the imagination of a large part of humanity; it shows them aesthetic lights and almost mystic insights.

Not only does chess (like many parts of mathematics) not need external, real-world revitalization, but, in fact, every now and then it spontaneously revitalizes itself. The most recent time that occurred was when retrograde chess analysis began to be studied seriously. (Sample problem: a chess position is given; you are to decide which side of the board White started from and whether Black had ever castled.) And here's a switch: not only was the real world not needed to revitalize chess, but, in fact, the life giving went the other way. Retrograde chess analysis challenged computer scientists with a new kind of problem, and it now constitutes a small but respectable and growing part of applied mathematics.

A revitalist might not be convinced by all this; he might point to the deplorable tendency of mathematics to become ultra-abstract, ultra-complicated, and involutedly ugly, and say that contact with the applications remedies that. The disease exists, that is well known, but, fortunately, so does nature's built-in cure. Several parts of mathematics have become cancerously overgrown in the course of the centuries; certain parts of elementary Euclidean geometry form a probably non-controversial example. When that happens, a wonderful remission always follows. Old mathematics never dies—what the Greeks bequeathed us 2500 years ago is still alive and true and interesting—but the outgrowths get simplified, their valuable core becomes integrated into the main body, and the nasty parts get sloughed off.

(Parenthetically: the revitalization argument could, in principle, be ap-

plied to painting, but so far as I know no one has applied it. Painting originates in the real world, it has been known to leave that world for realms of abstraction and complication that some find repulsive, but the art as a whole continues alive and well through all that.

A time argument is sometimes mentioned as a good feature of the contact of pure mathematics with applied. Example: if only pure mathematicians had paid closer attention to Maxwell, they would have discovered topological groups much sooner. Perhaps so—but just what did we lose by discovering them later? Would the world be better off if Rembrandt had been born a century earlier? What's the hurry?)

Whether contact with applications can prevent or cure the disease of elaboration and attenuation in mathematics is not really known; what is known is that many of the vigorous and definitely non-cancerous parts have no such contact (and probably, because of their level of abstraction, cannot have any). Current examples: analytic number theory and algebraic geometry.

When I say that mathematics doesn't *have* to be freshened by periodic contact with reality, of course I do not mean that it *must not* be: many of the beautiful concepts of pure mathematics were first noticed in the study of one or another part of nature. Perhaps they would not have been discovered without external stimulation, or perhaps they would—certainly many things were.

As far as the interaction between pure and applied mathematics is concerned, the truth seems to be that it exists, in both directions, but it is much stronger in one direction than in the other. For pure mathematics the applications are a great part of the origin of the subject and continue to be an occasional source of inspiration—they are, however, not indispensable. For applied mathematics, the pure concepts and deductions are a tool, an organizational scheme, and frequently a powerful hint to truths about the world—an indispensable part of the applied organism. It's the ant and the anteater again: arguably, possibly, the anteater is of some ecological value to the ant, but, certainly, indisputably, the ant is necessary for the anteater's continued existence and success.

What's next?

The most familiar parts of mathematics are algebra and geometry, but for the profession there is a third one, analysis, that plays an equally important role. Analysis starts from the concept of change. It's not enough just to study sizes and shapes; it is necessary also to study how sizes and shapes vary. The natural way to measure change is to examine the difference between the old and the new, and that word, "difference", leads in an etymologically straight line to the technical term "differential equation".

Most of the classical parts of applied mathematics are concerned with change—motion—and their single most usable tool is the theory and technique of differential equations.

Phenomena in the real world are likely to depend on several variables: the success of the stew depends on how long you cook it, how high the temperature is, how much wine you add, etc. To predict the outcome correctly, the variables must be kept apart: how does the outcome change when a part of the data is changed? That's why much of applied mathematics is inextricably intertwined with the theory of *partial* differential equations; for some people, in fact, the latter phrase is almost a synonym of applied mathematics.

Are great breakthroughs still being made and will they continue? Is a Shakespeare of mathematics (such as Archimedes or Gauss) likely to be alive and working now, or to be expected ever again? Algebra, analysis, and geometry—what's the mathematics of the future, and how will the relations between pure and applied develop?

I don't know the answers, nobody does, but the past and the present give some indications; based on them, and on the hope that springs eternal, I'll hazard a couple of quick guesses. The easiest question is about great breakthroughs: yes, they are still being made. Answers to questions raised many decades and sometimes centuries ago are being found almost every year. If Cantor, Riemann, and Poincaré came alive now, they would be excited and avid students, and they would learn much that they wanted to know.

In the foreseeable future discrete mathematics will be an increasingly useful tool in the attempt to understand the world, and . . . analysis will therefore play a proportionally smaller role.

Is there an Archimedes alive now? Probably not. Will there ever be another Gauss? I don't see why not; I hope so, and that's probably why I think so.

I should guess that in the foreseeable future (as in the present) discrete mathematics will be an increasingly useful tool in the attempt to understand the world, and that analysis will therefore play a proportionally smaller role. That is not to say that analysis in general and partial differential equations in particular have had their day and are declining in power; but, I am guessing, not only combinatorics but also relatively sophisticated number theory and geometry will displace some fraction of the many pages that analysis has been occupying in all books on applied mathematics.

Applied mathematics is bound to change, in part because the problems

change and in part because the tools for their solution change. As we learn more and more about the world, and learn how to control some of it, we need to ask new questions, and as pure mathematics grows, sloughs off the excess, and becomes both deeper and simpler thereby, it offers applied mathematics new techniques to use. What will all that do to the relation between the ant and the anteater? My guess is, nothing much. Both kinds of curiosity, the pure and the practical, are bound to continue, and the Socrates of 2400 years from now will probably see the difference between them as clearly as did the one 2400 years ago.

So, after all that has been said, what's the conclusion? Perhaps it is in the single word "taste".

A portrait by Picasso is regarded as beautiful by some, and a police photograph of a wanted criminal can be useful, but the chances are that the Picasso is not a good likeness and the police photograph is not very inspiring to look at. Is it completely unfair to say that the portrait is a bad copy of nature and the photograph is bad art?

Much of applied mathematics has great value. If an intellectual technique teaches us something about how blood is pumped, how waves propagate, and how galaxies expand, then it gives us science, knowledge, in the meaning of the word that deserves the greatest respect. It is no insult to the depth and precision and social contribution of great drafters of legislative prose (with their rigidly traditional diction and style) to say that the laws they write are bad literature. In the same way it is no insult to the insight, technique, and scientific contribution of great applied mathematicians to say of their discoveries about blood, and waves, and galaxies that those discoveries are first-rate applied mathematics; but, usually, applied mathematics is bad mathematics just the same.