Extracting waves and vortices from Lagrangian trajectories


Received 27 September 2011; revised 25 October 2011; accepted 25 October 2011; published 7 December 2011.

A method for extracting time-varying oscillatory motions from time series records is applied to Lagrangian trajectories from a numerical model of eddies generated by an unstable equivalent barotropic jet on a beta plane. An oscillation in a Lagrangian trajectory is represented mathematically as the signal traced out as a particle orbits a time-varying ellipse, a model which captures wavelike motions as well as the displacement signal of a particle trapped in an evolving vortex. Such oscillatory features can be separated from the turbulent background flow through an analysis founded upon a complex-valued wavelet transform of the trajectory. Application of the method to a set of one hundred modeled trajectories shows that the oscillatory motions of Lagrangian particles orbiting vortex cores appear to be extracted very well by the method, which depends upon only a handful of free parameters and which requires no operator intervention. Furthermore, vortex motions are clearly distinguished from free parameter trajectories shows that the oscillatory motions of Lagrangian trajectories from a numerical simulation in order to illustrate the possibility of accurately extracting and distinguishing vortex currents and wavelike motions. All relevant analysis software is freely distributed to the community as a part of a Matlab® toolbox, available at http://www.jmlilly.net.

1. Introduction

[2] Understanding the role of long-lived eddies in the global ocean circulation is a major topic in oceanographic research. Lagrangian floats and drifters constitute an invaluable platform for observing the relatively small spatial and temporal scales associated with such vortices, and indeed many dozens of publications have examined vortex properties from regional Lagrangian experiments [e.g., Armi et al., 1989; Flament et al., 2001; Lankhorst, 2006]. However, the fact that this problem remains unsolved is evidenced by the fact that large-scale studies either continue to rely on traditional subjective identification [e.g., Shoosmith et al., 2005], or else to focus on measures of the effect of vortices on trajectories rather than the properties of the vortices themselves [Griffa et al., 2008].

[3] Recently a new and general method has been developed [Lilly and Gascard, 2006; Lilly and Olhede, 2009a, 2009b, 2011], grounded in nonstationary time series theory, which permits the automated identification, extraction, and analysis of time-varying oscillatory features of unknown frequency—such as the signature of a particle trapped in a vortex or advected by a wave. Here the method is applied to Lagrangian trajectories from a numerical simulation in order to illustrate the possibility of accurately extracting and distinguishing vortex currents and wavelike motions. All relevant analysis software is freely distributed to the community as a part of a Matlab® toolbox, available at http://www.jmlilly.net.

2. Numerical Model

[4] For an idealized numerical model generating eddies as well as background variability, we choose an equivalent barotropic quasigeostrophic model of an initially unstable jet on a beta plane. The model integrates the equation for conservation of potential vorticity following the geostrophic flow

\[
\frac{\partial}{\partial t} + \mathbf{k} \cdot \nabla \Phi \times \nabla \left( \nabla^2 \Phi - \Phi/L_D^2 \right) + \beta y = 0
\]

(1)

where \( \Phi \) is the streamfunction, \( \beta \equiv df/d\theta \) is the derivative of the Coriolis frequency \( f \) with latitude, the parameter \( L_D \) is the Rossby radius of deformation, and \( \mathbf{k} \) is the vertical unit vector. The model is initialized at time \( t = 0 \) with an eastward jet of strength \( U \) and width \( Y \) having a profile, with \( \mathbf{i} \) being the eastward unit vector, given by

\[
|\mathbf{u}|_{0-} = \mathbf{i} \begin{cases} U \cos^2 \left( \frac{y}{2Y} \right) & |y|/Y \leq 1 \\ 0 & |y|/Y > 1 \end{cases}
\]

(2)

which corresponds to a maximum initial vorticity anomaly within the jet of \( \zeta_0 = \pi U/(2Y) \).

[5] Parameters are chosen to give a strong jet with a deformation radius that is small compared to the radius of the earth. The central latitude (\( y = 0 \)) at the jet axis is set to \( \theta = 45^\circ \) N, the jet width \( 2Y \) and deformation radius \( L_D \) are both 80 km, and the maximum initial velocity is 2.08 m s\(^{-1}\). These choices give a jet Rossby number \( Ro \equiv \zeta_0 f = 0.80 \), and a value of \( \beta L_D \zeta_0 \) of \( 1/(2\pi) \approx 1/60 \)—meaning that the jet relative vorticity anomaly is much larger than the change in planetary vorticity over one deformation radius, or over the jet width. This system is a convenient way...
of generating eddies, and is not intended to represent a particular oceanic current. The initial condition is set to freely evolve for 360 days with a time step of $5 \times 10^{-5}$ × 360 days ≈ 26 minutes in a domain of length $2\pi \times 400 \approx 2500$ km on each side. The model is seeded with 100 drifters that all initially lie along the $y = 0$ line, sampled every 10 time steps or 4.3 hours.

A snapshot of the model is shown in Figure 1, together with overlays of ellipses characterizing oscillatory Lagrangian variability from our subsequent analysis. Vortices are generated by barotropic instability and then tend to drift westward as well as meridionally due to nonlinear beta drift [e.g., Lam and Dritschel, 2001], with anticyclones propagating equatorward and cyclones propagating poleward. Dipole interactions are also seen, as captured by the red/blue pair of ellipses in the upper left quadrant of Figure 1, although these tend to eventually break down. An informative animation of snapshots such as the one shown in Figure 1 over the entire model run duration, but with the ellipses color coded by drifter number for visual clarity, is included as a part of the auxiliary material. Trajectories are shown in Figure 2a, with the dominating presence of vortex motions apparent as tightly looping or cycloidal curves reaching northwestward and southwestward.

3. Analysis Method

[7] A modulated elliptical signal, introduced by Lilly and Olhede [2010], is a time-varying oscillation in two dimensions. Such a signal is expressed in matrix form as

$$\mathbf{x}(t) = \begin{bmatrix} \cos \theta(t) & -\sin \theta(t) \\ \sin \theta(t) & \cos \theta(t) \end{bmatrix} \begin{bmatrix} a(t) \cos \phi(t) \\ b(t) \sin \phi(t) \end{bmatrix}$$

which is the parametric equation for a particle orbiting a time-varying ellipse with semi-major and semi-minor axes $a(t)$ and $b(t)$, where $a(t) > |b(t)| > 0$, and with its major axis inclined at an angle $\theta(t)$ with respect to the x-axis. The phase $\phi(t)$ gives the instantaneous position of the particle along the periphery of ellipse. The particle orbits the ellipse in the mathematically positive or negative direction according to the sign of $b(t)$.

[8] Details of the modulated elliptical signal, including conditions for associating a unique set of time-varying ellipse parameters to a given oscillatory signal $\mathbf{x}(t)$, are discussed by Lilly and Olhede [2010]. An important special case is that of a familiar pure sinusoidal oscillation in two dimensions; however, the model (3) is considerably more general. A practical constraint is that the ellipse properties should vary slowly compared with the timescale $2\pi/\mu \phi(t)$ over which the particle orbits the ellipse, in order that the subsequent analysis method have small errors [Lilly and Olhede, 2011].

[9] The modulated elliptical signal with slowly-varying ellipse parameters is a good model for the displacement signal of a particle trapped in an evolving vortex. This class of signals includes Lagrangian displacements due to steady circular vortex solutions, steadily strained or sheared elliptical anticyclones [Ruddick, 1987], and low-frequency periodic oscillations of an elliptical shallow water vortex [Young, 1986; Holm, 1991], all observed with instruments that may be experiencing a drift through the vortex in addition to the vortex currents themselves.

[10] Ellipse size, shape, and frequency are usefully characterized as follows. The ellipse shape is described by the eccentricity $\epsilon(t) = \sqrt{1 - b^2(t)/a^2(t)}$. The geometric mean radius and geometric mean velocity are defined as

$$R(t) = \sqrt{\bar{a}(t)|\bar{b}(t)|}, \quad V(t) = \text{sgn} \{b_u(t)\} \sqrt{a_u(t)|b_u(t)|}$$

where the latter quantity is found by writing the velocity $\mathbf{u}(t) = \frac{\partial}{\partial t} \mathbf{x}(t)$ as a time-varying ellipse of the form (3) but with a different set of ellipse parameters, denoted $a_u(t), b_u(t)$, etc.; this may be accomplished algebraically from the parameters of $\mathbf{x}(t)$, see Appendix E of Lilly and Gascard [2006]. The ratio of the two quantities $V(t)$ and $R(t)$ defines a frequency $\omega(t) = V(t)/R(t)$ which we call the geometric frequency.

[11] A Lagrangian trajectory can then be represented as the sum of a number $M$ of different oscillatory displacement signals $\mathbf{x}^{[m]}(t)$, each of the form (3), plus a residual:

$$\mathbf{x}(t) = \sum_{m=1}^{M} \mathbf{x}^{[m]}(t) + \mathbf{e}(t).$$

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1Auxiliary materials are available in the HTML. doi:10.1029/2011GL049727.
The residual signal $\mathbf{e}(t)$ includes the turbulent background flow, as well as any non-oscillatory component such as a mean flow or the systematic self-propagation tendency of a vortex. The oscillatory signals may be of finite duration and may be overlapping in time, so that zero, one, or more than one such signal may be present at each moment.

[12] The problem is then to estimate the oscillations $\mathbf{x}^{(m)}(t)$ given an observed trajectory $\mathbf{x}(t)$, and from these estimates to characterize the $M$ different ellipse parameters $a^{(m)}(t)$, $b^{(m)}(t)$, $\theta^{(m)}(t)$, $\phi^{(m)}(t)$, and so forth. This is accomplished using a method called “multivariate wavelet ridge analysis” [Lilly and Olhede, 2009a, 2009b, 2011], leading to estimates $\mathbf{\hat{x}}^{(m)}(t)$ of the $M$ modulated oscillations in each trajectory; here the subscript “$\psi$” indicates the choice of filter or wavelet $\psi(t)$ used in the analysis. A discussion of the basic idea and implementation of the method, together with details of the parameter settings used here, are provided in a technical text file that is included in the auxiliary material.

[13] From the method, we obtain estimates of ellipse properties at each moment. As an example, observe the good agreement in Figure 1 between the ellipses, inferred non-locally from individual particles, and the Eulerian structures in the potential vorticity field.

4. Results

[14] The results of this analysis are shown in Figures 2 and 3. Subtracting the sum over all $M$ estimated oscillations in each time series $\mathbf{x}(t)$ leads to an estimate $\mathbf{\hat{e}}(t)$ of the

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**Figure 2.** (a) All Lagrangian trajectories from the unstable barotropic jet simulation, dispersing from their initial location at $y = 0$, with each trajectory in a different color. (b) The residual curves from the wavelet ridge analysis. (c, d) Snapshots of ellipses corresponding to the ellipse properties estimated from the wavelet ridge analysis. In Figures 2c and 2d, highly eccentric ellipses with $\varepsilon > 0.95$—typically very low-frequency signals—are plotted with a time step of $1/2$ the estimated period, while other ellipses are plotted with the time step of every 3 estimated periods. The highly eccentric ellipses are plotted in Figure 2c in green, while the remaining anticyclonically rotating ellipses and cyclonically rotating ellipses are plotted in blue and red, respectively. Finally in Figure 2d, the ellipses are colored according to $\log_{10}$ of the estimated geometric period $2\pi/\varepsilon(t)$, measured in days, as indicated in the color bar.
non-oscillatory background flow, Figure 2b. Comparison with the original time series, Figure 2a, shows that the tightly looping motions associated with vortices appear to have been nearly completely removed. What remains behind is observed to consist of curving but disorganized motions, together with systematic motion associated with the jet and with eddy drift. Even cycloidal features in Figure 2a, typically low-frequency motion of a particle on the far flank of an eddy, are largely removed. The ability to perform such a separation on this relative large set of drifters, with no intervention by the analyst, by itself constitutes a technical breakthrough.

Instantaneous ellipses associated with the estimated oscillatory signals are shown in Figures 2c and 2d. Coloring the ellipses according to their degree of eccentricity and sense of rotation in Figure 2c reveals nearly exclusively cyclonic motions (red) on the poleward side of the jet and anticyclonic motions (blue) on the equatorward side. The separation of vortices by their polarity in this system is due to nonlinear beta drift acting on eddies generated within the jet core during the initial adjustment. Close inspection reveals some very small circles of opposite color inside some of the eddies; as discussed in the technical appendix, these represent weak, low-frequency signals, likely associated with the presence of exterior opposing vorticity anomalies. Another type of variability, nearly linear in polarization and oriented meridionally, is observed along the jet axis (green).

A time scale distinction between these two different types of motions can be seen in Figure 2d, where the color coding represents the instantaneous oscillation period $2\pi/\omega(t)$ as deduced from the geometric frequency $\omega(t)$. The highly eccentric motions along the jet axis are seen to be an order of magnitude lower in frequency than the vortex motions. The former arise as particles in the jet are swept eastward through meanders caused by the deflection of the jet axis by low-frequency Rossby wave variability, as is readily apparent in Animation S1 in the auxiliary material.

A more detailed view of the properties of the estimated elliptical signals is found in the distribution plots on the radius/velocity plane of Figure 3. A histogram on the geometric radius $R(t)$/geometric velocity $V(t)$ plane occupied by all estimated oscillatory signals from all 100 trajectories, with a logarithmic color axis. (b) The median eccentricity $\varepsilon$ in each bin. In Figures 3a and 3b, the sloping gray lines correspond to $\pm \zeta_0/2$, the geometric frequency value corresponding (for an eddy in solid-body rotation) to the maximum initial jet relative vorticity anomaly. The black lines are plus or minus the maximum and minimum cutoff frequencies in the analysis.
maximum vorticity anomaly of the initial unstable jet profile (gray line); because under the assumption of solid-body rotation we have \( \zeta = 2V/R \), this occurs at a frequency of \( \zeta/2 = 0.40f \).

[18] This vortex profile pattern emerges primarily through the superposition of a number of different Lagrangian particles at different locations in different eddies. On the anticyclonic side the pattern is less evident, since by chance, more particles have ended up in cyclonic eddies compared to anticyclonic eddies in this simulation; this asymmetry is a reminder of the well-known slow convergence of Lagrangian statistics owing to long-term particle trapping [e.g., Pasquero et al., 2002].

[19] The low-frequency motions are evident as the symmetric maxima in Figure 3a around the horizontal line \( V' = 0 \). The median eccentricity \( \varepsilon \) in each bin, Figure 3b, shows that the motions in the vortex profile are nearly circular in polarization, while the low-frequency motions are nearly linear, confirming that these two regions on the \( R/V \) plane correspond to the two types of features seen in Figures 2c and 2d. An important point is that the nearly circular vortex motions and the highly eccentric low-frequency motions occupy almost non-overlapping regions of the radius/velocity plane, apart from very large-scale (say \(~50 \text{ km} \) radius) and low-frequency motions, which may either be generated by jet meander or by circular oscillatory motions on the far flank of a vortex.

5. Conclusions

[20] This paper has applied a new analysis method—multivariate wavelet ridge analysis—to the identification of oscillatory motions in Lagrangian trajectories from a numerical simulation of an unstable jet. It is shown that vortex motions can be effectively extracted from the trajectories and described locally in terms of their time-varying frequency content and ellipse geometry. Meridional meandering of the jet axis constitutes another strong oscillatory signal in this model, but these motions are clearly distinguished from the vortex motions on account of their much lower frequency, nearly linear shape polarization, and different location in radius/velocity space. The method is therefore able to unravel the superposition of different processes in individual trajectories, making possible the investigation of these separate processes in isolation from one another. These results serve as validation supporting the application of this method to large-scale observational studies. Not addressed here, but left to the future, are a detailed treatment of stochastic errors, the impact of measurement noise, comparison between the Eulerian and Lagrangian perspectives, and a consideration of method performance for other types of motions such as inertial oscillations or baroclinic instability waves.

[21] Acknowledgments. The work of J. M. Lilly was supported by awards #0751697 and #0849371 from the Physical Oceanography program of the United States National Science Foundation. The work of S. C. Olhede was supported by award EP/005250/1 from the Engineering and Physical Sciences Research Council of the United Kingdom.

[22] The Editor thanks the two anonymous reviewers for their assistance in evaluating this paper.

References


J. M. Lilly, NorthWest Research Associates, 4118 148th Ave. NE, Redmond, WA 98052, USA. (lilly@nwra.com)
S. C. Olhede, Department of Statistical Science, University College London, Gower Street, London WC1E 6BT, UK.
R. K. Scott, Department of Applied Mathematics, University of St Andrews, St Andrews KY16 9SS, UK.