On a new way of writing the Navier-Stokes equation. The Hamiltonian formalism

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On a new way of writing the Navier-Stokes equation.
The Hamiltonian formalism

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1. Let \( \gamma \) be a vector field on \( \mathbb{R}^n \) and

\[
\psi(r) = \int G(r - r') \psi(r') d^n r',
\]

\[
G(r) = c_n^{-1} |E| r_n - n(r \times r) |r|^{-n-2},
\]

where \((-c_n)\) is the area of the unit sphere in \( \mathbb{R}^n \). We set \( \omega = \gamma dr, \gamma = \gamma(t, r), r \in \mathbb{R}^n \) and consider the equation

\[
\psi_t = L_\psi \omega - \nu \Delta \omega,
\]

where \( L_\psi \) is the Lie derivative and \( \nu > 0 \) is a constant.

**Proposition 1.** If \( \gamma \) satisfies (2), then \( \psi \) satisfies the Navier-Stokes equation

\[
\psi_t + (\psi \nabla) \psi = -\text{grad} p + \nu \Delta \psi, \quad \text{div} \psi = 0.
\]

**Proposition 2.** If \( \gamma \) satisfies (2), \( \psi = \psi + \text{grad} \varphi \), then \( \varphi \) satisfies the equation

\[
\varphi_t + (\psi \nabla) \varphi = p - \frac{\nu^2}{2} + \nu \Delta \varphi.
\]

**Proposition 3.** If \( \psi \) satisfies the Navier-Stokes equation and \( \varphi \) satisfies (3), then \( \gamma = \psi + \text{grad} \varphi \) satisfies equation (2).

2. Our aim is to draw attention to the fact that for \( \nu = 0 \) equation (2) is a Hamiltonian system with Hamiltonian

\[
H = \frac{1}{2} \int \int G(r - r') \gamma(r) \gamma(r') d^n r d^n r',
\]

and Lie-Poisson brackets

\[
\{ \gamma(a), \gamma(b) \} = \gamma((a \nabla)b - (b \nabla)a),
\]

where

\[
\gamma(a) = \int \gamma(r) a(r) d^n r.
\]

In fact a direct computation shows that the equation

\[
\dot{\gamma} = \{ \gamma, H \}
\]

is equivalent to (2) with \( \nu = 0 \).

The Euler equation of an incompressible fluid is usually written on the quotient space of the space of vector fields by the subspace of gradient fields (see [2]). It is easy to see that this way of writing it is obtained when equation (5) is reduced to the quotient space, since

\[
H = \frac{1}{2} \int v \gamma d^n r = \frac{1}{2} \int v^2 d^n r.
\]

3. The convenience of writing the Euler equation in the new form (5) becomes apparent in the Lagrangian approach. Let \((r(\xi), \gamma(\xi), \xi \in \mathbb{R}^n)\) be the Lagrangian variables. Then

\[
\dot{r} = \frac{\delta H}{\delta \gamma(\xi)}, \quad \dot{\gamma} = -\frac{\delta H}{\delta r(\xi)},
\]

\[
H = \frac{1}{2} \int \int G(r(\xi) - r(\xi')) \gamma(\xi) \gamma(\xi') d^n \xi d^n \xi'.
\]

It follows from (6) that if \( \gamma = \gamma(t, r) = \gamma(\xi) \) for that value of \( \xi \) for which \( r = r(\xi) \), where \((r(\xi), \gamma(\xi))\) is a solution of the system (6), then \( \gamma = \gamma(t, r) \) is a solution of (5). The "Lagrange-Euler map" just described (see [3]) takes the canonical Poisson brackets for the Lagrangian variables to the Lie-Poisson brackets for the Euler variables and the Hamiltonian \( H \) in (6) to the Hamiltonian \( H \) in (4).
Carrying out an argument analogous to that on p.19 of [2] (that is, by introducing a set of discrete vortex dipoles), we have

\[ \gamma = \sum_{\alpha=1}^{N} \gamma_{\alpha} \delta (r - r_{\alpha}), \quad H = \frac{1}{2} \sum_{\alpha \neq \beta} G (r_{\alpha} - r_{\beta}) \gamma_{\alpha} \gamma_{\beta}, \]

where the \((r_{\alpha} = r(\xi_{\alpha}), \gamma_{\alpha})\) are canonical variables, and

\[ \{r_{\alpha}, \gamma_{\beta}\} = \delta_{\alpha \beta} E, \quad \{r_{\alpha}, r_{\beta}\} = \{\gamma_{\alpha}, \gamma_{\beta}\} = 0. \]

For \(n = 3\) the Hamiltonian system with such an \(H\) is the Hamiltonian system of the Roberts equations (without taking self-action into account (see [4] - [6]). We note that the Hamiltonian system with Hamiltonian \(H\) in (7) is integrable for \(N = 2\) and \(\gamma_1 + \gamma_2 = 0\) for any \(n > 2\); here for any \(n > 2\) a collapse is possible for it (that is, a confluence of two dipoles after a finite time). This was shown in [6] for \(n = 3\). We note one other integrable case: \(r_1 - r_2, \gamma_1, \gamma_2\) lying on a straight line. Here collapse is also possible.

4. We make a number of concluding remarks. For \(\nu = 0\) equation (2) is a "strengthened" version of Kelvin's theorem on the circulation of velocity. The equation for the vortices obtained from (2) by applying the exterior differential is

\[ \partial_t \omega^3 = - L_\nu \omega^3 - \nu \Delta \omega^3. \]

Hence it follows that for \(n = 3\) and \(\nu = 0\)

\[ \partial_t \omega^3_{y_1} = - L_1 \omega^3_{y_1}. \]

since \(\omega^3_{y_1} = \omega^3 \wedge \omega_1^3\). Here \(\Omega = \text{rot} \gamma = \text{rot} \nu\). Consequently the spirality \(\gamma \Omega\) is transferred by the fluid particle, which gives the spirality integrals \(\int F(\gamma \Omega) d^3r\). This kind of integral has been detected by Gordin and Petviashvili for a non-viscous compressible barotropic fluid [1]. They introduced \(\gamma\) as the vector-potential for \(\Omega\) with "spiral gauge":

\[ \partial_t \gamma = - \text{grad}(\nu \gamma) + \gamma \times \Omega. \]

Formula (1) does not hold for a compressible fluid and consequently the Hamiltonian does not have an interpretation analogous to that given in part 2; however, the above relation can be written in the form

\[ \partial_t \omega^1_1 = - L_\nu \omega^1_1. \]

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References


