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Almost horizontal turbulence

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Abstract. In this note an attempt is made to create the necessary prerequisites for the development of a theory of ‘almost horizontal turbulence’. A system of simplified equations of ‘almost horizontal motions’ is proposed. These equations are affinely invariant, namely, one can independently modify the vertical and horizontal scales. It is shown that the simplification used does not prevent us from obtaining a correct theory of internal waves whose frequencies are significantly less than the Brent–Väisälä frequency.

1. The mathematical model of ‘two-dimensional turbulence’ can in fact be applied to the motion of layers of some finite thickness. For instance, it is assumed that two-dimensional considerations of this kind lead to a reasonable model of formation of meanders with horizontal sizes of hundreds of kilometers on a flow involving a layer of hundreds of meters, with subsequent disintegration of these meanders into vortices gradually decreasing in size to several kilometers. If one has in mind a layer of constant density with intensive vertical mixing and if this layer can slide rather freely along the underlying layer with a strong steady-state stratification, then these concepts seem sufficiently justified.

However, data have gradually accumulated about the detailed structure of vertical profiles of temperature, salinity, density, and velocity in layers with steady-state stratification, together with data about turbulized ‘pancakes’ and the
non-turbulized interlayers (in the sense of small-scale three-dimensional turbulence) that separate the pancakes, data forcing us to assume that in this case, at the corresponding scales, the horizontal displacements of rather thin layers are independent of one another to a great extent. An ‘almost horizontal turbulence’ must occur such that its structure functions,

\[ D_{ij}(\Delta \vec{x}) = \Delta v_i \Delta v_j, \]

increase under a vertical displacement much more rapidly than under a horizontal displacement.

In this note we make an attempt to create the necessary prerequisites for the development of a theory of an ‘almost horizontal turbulence’ of this kind. We propose a system of simplified equations of ‘almost horizontal motions’, and note that these equations are affinely invariant: one can independently modify the vertical and horizontal scales. It is shown that our simplification does not prevent us from obtaining a correct theory of internal waves whose frequencies are significantly less than the Brent–Väisälä frequency.

The proposed simplified system of equations is of the form

\[
\begin{align*}
\frac{du}{dt} + fv + \frac{\partial p}{\partial x} &= \Phi_x, \\
\frac{dv}{dt} - fv + \frac{\partial p}{\partial y} &= \Phi_y, \\
\frac{\partial p}{\partial z} - b &= 0, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, \\
\frac{db}{dt} &= \Phi_b.
\end{align*}
\]

Here, as usual, \( p \) stands for the deviation of the pressure from the standard value \( P = -z \) (we assume that the standard density \( \rho_0 \) is equal to one) and \( b \) is the ‘floatage’. The quantities \( u, v, w, p, \) and \( b \) enter the equations in the form of their mean values over the volume, which makes it possible to adopt the assumption that the motion is ‘almost horizontal’ (expelling the term \( \frac{dw}{dt} \) from (1.3) and including the influence of viscosity and diffusion in the terms \( \Phi_x, \Phi_y, \Phi_b \), which have the meaning of interaction with the ‘micro-component’ of our quantities).

The system of equations (1.1)–(1.5) is invariant with respect to the change of variables

\[
\begin{align*}
x &= \lambda x', \\
y &= \lambda y', \\
u &= \lambda u', \\
v &= \lambda v', \\
p &= \lambda^2 p', \\
z &= \lambda_1 z', \\
w &= \lambda_1 w', \\
b &= \lambda^2 \lambda_1^{-1} b',
\end{align*}
\]

where \( \lambda \) and \( \lambda_1 \) are chosen independently. It should be noted that the Richardson number

\[ \text{Ri} = -\frac{\partial b}{\partial z} \left( \frac{\partial u}{\partial z} \right)^2 \]

is also preserved under this change of variables.
2. To control the character of the above simplifications, we shall now construct the corresponding linear theory of internal waves. Let us represent \( p \) and \( b \) as the sums of their mean values \( \bar{p} \) and \( \bar{b} \), which depend only on \( z \), and the deviations \( p' \) and \( b' \) from the mean values, and write out the linearized system for \( u, w, p', \) and \( b' \), where the motion takes place in the \((x,y)\) plane:

\[
L_1 = \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0, \tag{2.1}
\]

\[
L_2 = \frac{\partial p'}{\partial z} - b' = 0, \tag{2.2}
\]

\[
L_3 = \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0, \tag{2.3}
\]

\[
L_4 = \frac{\partial b'}{\partial t} + w \frac{\partial \bar{b}}{\partial z} = 0. \tag{2.4}
\]

It follows from (2.1)–(2.5) that

\[
\frac{\partial^3}{\partial t \partial x \partial z} L_1 - \frac{\partial^3}{\partial t \partial x^2} L_2 + \frac{\partial^3}{\partial t^2 \partial z} L_3 - \frac{\partial^2}{\partial t \partial z^2} L_4 = \frac{\partial^4 w}{\partial t^2 \partial z^2} - \frac{\partial \bar{b}}{\partial z} \frac{\partial^2 w}{\partial x^2} = 0;
\]

setting \( -\frac{\partial \bar{b}}{\partial z} = N^2 \), we obtain

\[
\frac{\partial^4 w}{\partial t^2 \partial z^2} + N^2 \frac{\partial^2 w}{\partial x^2} = 0. \tag{3}
\]

This equation differs from the full equation

\[
\frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right) + N^2 \frac{\partial^2 w}{\partial x^2} = 0 \tag{3a}
\]

in the absence of the term \( \frac{\partial^2 w}{\partial x^2} \), which is small in comparison with the term \( \frac{\partial^2 w}{\partial z^2} \) for ‘almost planar’ waves.

Setting \( w = \alpha(z) \exp i(\kappa x - nt) \),

we obtain

\[
n^2 \alpha'' + N^2 k^2 \alpha = 0. \tag{4}
\]

In the study of multilayer almost planar turbulence, waves of higher modes can be of importance. To understand the structure of these waves, one considers perturbations of the form

\[
w = \gamma \exp i(kx + mz - nt) = \gamma E. \tag{5}
\]

By the equations (2) we have

\[
w_z = im \gamma E, \quad u_x = -im \gamma E, \quad u = \frac{m}{k} \gamma E,
\]

\[
u_t = -im \frac{k}{m} \gamma E, \quad w_t = -i \gamma E, \quad w_x = i k \gamma E. \tag{5.1}
\]
In our case the dispersion relation has the following simple form:

\[ nm = Nk. \]  

(6)

The ‘almost planar’ character of waves requires the conditions

\[ k \ll m, \quad n \ll N, \]

which explains the validity of the simplification (6) in comparison with the usual expression

\[ nm = \sqrt{N^2 - n^2 k}. \]  

(6.1)

Let us also consider conditions for the validity of the linearization of the equations. To this end, using (5) and (5.1), we write out the orders of individual terms in the total derivatives \( \frac{du}{dt} \):

\[
\begin{vmatrix}
\partial u \\ \partial t \\
\frac{nm}{k}\gamma \\
\frac{m^2}{k}\gamma
\end{vmatrix}
\begin{vmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial z} \\
\frac{\partial u}{\partial z} \\
\frac{\partial u}{\partial z}
\end{vmatrix}
\]

We see that the non-linear terms are small in comparison with \( \frac{\partial u}{\partial t} \) if

\[ m\gamma \ll n. \]  

(7.1)

Using the fact that the amplitude of the vertical displacements has order \( A \sim \frac{\gamma}{n}, (6) \), we can represent (7.1) in the form

\[ A \ll L_z = \frac{2\pi}{m}. \]  

(7)

However, this is quite natural, namely, the role of the non-linear effects depends on the ratio of the amplitude of the vertical displacements to \( L_z \) rather than on the ‘steepness’ of the waves.

The study of ‘gently sloping’ but essentially non-linear waves seems to be a very interesting problem. It is unclear how much the simplification of the equations connected with our ‘almost horizontal’ property of the motions helps in this study. However, the remark about the affine invariance of the equations seems to be quite important in view of the absence of a mathematical theory for generalizing the experimental data. For instance, this invariance enables one to assume that, under some mechanism for increasing the energy of the wave motion, the ‘Umklapp’ or ‘breaking’ of waves can obtain the character of a ‘piling up’ of very thin layers one over another if \( L_z/L_x \) is small.
3. Assuming that \( v = w = 0 \) and that the quantities \( u, p, \) and \( b \) depend only on \( y \) and \( z \) and neglecting the terms \( \Phi_x, \Phi_y, \) and \( \Phi_b, \) we obtain from the equations (1) the standard equations of geostrophic flow

\[
-fu + \frac{\partial p}{\partial y} = 0, \quad (8.1)
\]

\[
\frac{\partial p}{\partial z} - b = 0, \quad (8.2)
\]

which implies that

\[
f \frac{\partial u}{\partial z} = \frac{\partial b}{\partial y}, \quad (8.3)
\]

a corollary no less known. The Richardson number becomes

\[
\text{Ri} = -f^2 \frac{\partial b}{\partial z} \left( \frac{\partial b}{\partial y} \right)^2. \quad (9)
\]

When studying multilayer turbulence, one must take into account the natural ratio of the horizontal and vertical scales,

\[
\frac{L_x}{L_z} = \frac{\partial b}{\partial x} / \frac{\partial b}{\partial y},
\]

which is usually very large.

At the present time it is hard for me to say whether or not the way of constructing a somewhat speculative concept of locally homogeneous and horizontally isotropic almost planar turbulence on this basis is fruitful. However, one should probably try this way.

4. Let us write out the equations for small perturbations of the geostrophic flow.\(^1\)

Waves with \( f \) taken into account. The dispersion relation has the form

\[
D = \begin{vmatrix}
-in & -f_0 & k/m \\
+f & -in & 0 \\
0 & 0 & im \\
0 & 0 & 0
\end{vmatrix} = -\left(n^2 - f^2\right)nm + N^2n \frac{k^2}{m} = 0,
\]

which implies that

\[
n^2 = f^2 + N^2 \frac{k^2}{m^2},
\]

\[-fa + inb = 0,\]

\[-N^2c + ind = 0,\]

\[
|a| = \frac{m}{k} |c|, \quad |b| = \frac{f}{n} |a| = \frac{fm}{nk} |c|.\]

\(^1\)Below we present the original calculations of Kolmogorov, which were regrettably not completed.
5. As the perturbations treated above develop, a ‘multilayer meandering’ must occur, which certainly cannot be subject to the linear theory. It is possible that a mechanism of refinement of the meanders thus arising sometimes acts; this mechanism can be predicted by some concept of ‘horizontal turbulence’ with transmission of energy or a vortex from some scales to others. However, it is natural to think that the interaction of layers located one above another must also play a large role. Under certain conditions three-dimensionally turbulized ‘pancakes’ can occur, even for a large Richardson number of the mean flow.

The next task, which is not very complicated, is a perturbation analysis according to the linear theory in the case \( \frac{\partial b}{\partial y} \neq 0 \). If a steady-state stratification is attained for different values of the cross-sectional coordinate \( y \) due to different combinations of temperature and salinity, then for a thin-layer meandering the effects must occur that have recently attracted such great attention.

For forming qualitative ideas concerning these processes, considerations related to the above affine invariance of the equations of almost planar motions must play some role. However, these equations lose their applicability when a turbulent vertical exchange takes place between the thin layers and in scales in which the Reynolds number of the thin layers can no longer be assumed to be inessential. Therefore, the averaging scale postulated from the very beginning must be assumed to be different in different situations. As we have seen above, the temporal scales of applicability of the concept thus developed must also be sufficiently large, because when we simplified we lost the condition \( n < N \) for the internal waves.