QUADRATIC DIFFERENTIAL SYSTEMS

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by

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Section 1

INTRODUCTION

The qualitative theory of differential equations is concerned with the global character of the set of solutions of a system

\[ \dot{x} = f(x) \]  \hspace{1cm} (1)

defined on an appropriate manifold. In recent years this theory has been greatly developed in an abstract setting which suppresses the notion of differential equation and concentrates upon the orbital structure of transformation groups acting on manifolds. The great beauty of this "topological dynamics" and the impressive progress resulting from its abstract viewpoint are indisputable. But together they have virtually eclipsed an important body of general problems remaining unsolved in the "concrete dynamics" concerned with translating explicit knowledge about the function or vector field \( f \) into equally explicit knowledge about the set of solutions of (1).

For this concrete aspect of qualitative theory, it is not enough to know what kinds of solutions are possible when \( f \) belongs to a general class of functions; methods are needed to determine which kinds do in fact exist when \( f \) is a given function.

Poincaré, whose 1894 memoir is often regarded as the starting point of modern qualitative theory, had a clear view of this division of the subject. On the one hand, he launched topological dynamics by solving one of its fundamental problems on \( S^1 \): every trajectory not a rest point or a closed cycle has limit sets which are one of these or a generalized cycle comprised of rest points joined by separatrices. On the other hand, he saw clearly the importance of the fundamental concrete problem to which the present paper contributes: Given, on the manifold \( E^n \) endowed with cartesian coordinates, a system (1) in which the components of \( f(x) \) are polynomial functions of
the components of \( x \), to decide the trajectory structure from the polynomial coefficients. Though topological dynamics has gone far beyond Poincaré's original result on \( S^2 \), this problem on \( E^n \) remains unsolved today for nonlinear systems, even in the case of \( E^2 \), where it subsumes the famous "problem of limit cycles."

The present work considers a simple aspect of this fundamental problem for polynomial systems on \( E^n \): The polynomials are restricted to be quadratic and \( t = 0 \) structural property investigated is boundedness.

The system

\[
\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij} x_j + \sum_{j,k=1}^{n} b_{ijk} x_j x_k \quad i = 1, \ldots, n
\]  

will be called weakly bounded if the solution issuing from each initial point of \( E^n \) can be continued to the entire positive \( t \)-axis, bounded if each positive half-trajectory is bounded in \( E^n \), and uniformly bounded if the bound is uniform for initial points in an arbitrary fixed compact subset of \( E^n \).

If the supremum of future times \( t \) to which the solution from a given initial point can be continued is called the escape time for that point, a weakly bounded system is described as one with no points of finite escape time. The problem of characterizing the weakly bounded polynomial systems is unsolved except in the trivial (linear) case. The attitude of topological dynamics toward this problem is to ignore it by concentrating attention on trajectories as geometric sets along which a change of local time suffices to eliminate the question. While interesting matters are thereby suppressed, there is little doubt that the characterization of bounded systems is the more important problem. Aside from the importance of boundedness in physical applications, the bounded systems are those in which topological dynamics leads us to expect interesting structure; all trajectories have compact positive limit sets and cannot simply escape to infinity. In \( E^2 \), for
example, the two systems

\[ \begin{align*}
\dot{x}_1 &= x_2 + \lambda x_2 (1 - x_1^2 - x_2^2) \\
\dot{x}_2 &= -x_1 + \lambda x_2 (1 - x_1^2 - x_2^2) \\
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \lambda x_2 - \lambda x_1^2 x_2 
\end{align*} \]

\( \lambda > 0 \)

(3)_1

(3)_2

have \((0,0)\) as a rest point which is unique and unstable. The knowledge that they are bounded therefore implies the existence of a limit cycle. This is already obvious from the polar coordinate form of the classical textbook example (3)_1, of course, but is less than obvious for the van der Pol system (3)_2.

Various devices are used classically to establish the boundedness of special systems. The existence of an integral \( F(x) \) such that each member of the family of hypersurfaces \( F(x) = \text{constant} \) is compact is one of them, typified by the system \( \dot{x} = F(x) x \), where \( F(x) \) is any skew symmetric \( nxn \) matrix with elements which are polynomials in the \( x_1 \) and \( F(x) = r = \sqrt{x_1^2 + \ldots + x_n^2} \). Another idea often made the basis of a boundedness proof may be roughly described as "asymptotic repulsion from infinity," and can be illustrated by the examples (3) above. Both these examples have the form \( \dot{x} = f_2(x) + f_3(x) \), where \( f_2 \) denotes the linear and \( f_3 \) the cubic terms from the right hand members. Hence the homogeneous cubic system \( \dot{x} = f_3(x) \) may be thought of as a "far field" approximation or asymptotic version of the system for large \( \|x\| \). In the case of (3)_1 the quartic form \( (x, f_3(x)) \) (inner product notation) is negative definite and boundedness follows from the fact that \( f < 0 \) on every sufficiently large sphere \( r = \text{constant} \), independently of any particular properties of the linear part \( f_1 \). For (3)_2, the situation is not quite so clear cut, since \( (x, f_3(x)) = -\lambda x_1^2 x_2^2 \) is only negative semi-definite, and the
argument must be modified to show that the particular linear terms do not permit solutions to become unbounded because of the lines on which \( \dot{x} = 0 \). But in either case, it seems fair to ascribe boundedness to the general idea of repulsion from infinity for the asymptotic system. In a quadratic system \((2)\), now abbreviated

\[
\dot{x} = f_1(x) + f_2'(x),
\]

the idea generally does not work. A homogeneous quadratic system \( \dot{x} = f_2(x) \) cannot exhibit such repulsion because of symmetry, or parity: since \( f_2(x) = f_2(-x) \), each solution \( x(t) \) can be paired with its "backwards-negative," \(-x(-t)\), which is also a solution, and if one goes to infinity as \( t \) decreases, the other goes to infinity as \( t \) increases. (This circumstance makes the boundedness problem for \((2)\) slightly more subtle than for cubic systems like \((3)\), and hence may account for the apparent absence from the literature of examples of plane quadratic systems which share with \((3)\) the properties of boundedness and possession of a rest point both unique and unstable.)

The following example shows that the repulsion idea does not always fail for quadratic systems. Let \( f_1(x) = Ax \) where \( A \) is a negative definite symmetric matrix, and let \( f_2(x) = B(x)x \) where \( B(x) \) is a skew symmetric matrix with linear homogeneous forms for elements. Then the system \((4)\) clearly exhibits repulsion from infinity, since \( \dot{x} = (x, \dot{x}) = \langle x, Ax \rangle < 0 \) for \( x > 0 \). Although the quadratic terms may dominate the magnitude of the vector field for large \( r \), their radial component vanishes identically, leaving the sign of \( \dot{x} \) to be determined by the linear terms alone. A far stronger conclusion than boundedness ensues: the origin is asymptotically stable in the large, independent of any particular structure of the matrix \( B(x) \) other than its skew symmetry.

This example is a clue which leads to a key idea for constructing bounded quadratic systems. Rather than negative definite, suppose \( A \) indefinite,

\[1.4\]

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so that $\mathbb{R}^n$ has both a "contraction" cone, where $rf = (x, Ax) < C_1$, and 
an "expansion cone," where $(x, Ax) > 0$. Now consider the "asymptotic" homogeneous quadratic system, which defines a flow on each sphere $\|x\| = \text{constant},$

$$\dot{x} = B(x)x;$$  \hspace{1cm} (5)

by homogeneity, it suffices to restrict to the unit sphere $\|x\| = 1$. If 
we suppose, for example, that this flow has all of its positive limiting 
sets in that part of the unit sphere $\|x\| = 1$ which belongs to the con-
tration cone, then the heuristic ingredients for boundedness of $(4)$ are 
clearly present: A solution which remains long in the expansion cone must 
ultimately be "turned" into the contraction cone by the dominance of the 
transverse quadratic component of the field for large $\|x\|$. 

Our first object in this paper is to prove a theorem which makes this 
heuristic idea precise. A system $(4)$ for which $f_2$ defines a field without 
radial component--i.e., for which the ternary form $(x, f_2(x))$ vanishes, 
identically--will be called "asymptotically transverse," or an AT system, 
and the corresponding system $(5)$ will be called an HT, or "homogeneous transverse," system. AT systems arise, in particular, as simplified 
analogs of the Navier-Stokes equations, called "models of turbulence." 

Roughly speaking, the $x_i$ correspond to Fourier components of the spatial 
distribution of turbulent velocities; the expansion cone corresponds to 
unstable spatial modes of flow whose energy tends to increase ("large 
eddies"), while the contraction cone corresponds to dissipative modes 
("small eddies," upon which viscous dissipation is effective); the ortho-
genality condition reflects the circumstance that the quadratically nonlinear 
terms do not contribute to the growth and decay of energy, but only provide 
the coupling which transfers energy between modes ("big eddies generate small 
eddies"). Viewed from this analogy, our boundedness theorem for AT systems 
represents one form of abstraction related to the mechanism of turbulent dis-
sipation.

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Because the respective symmetries and orders of magnitude of the functions \( f_1 \) and \( f_2 \) of (4) play the essential role, a number of variations and generalizations of the idea are suggested (for example, to other systems admitting an analogous decomposition, to a finite point instead of infinity, to other integrals for system (5), to function spaces and so on), but we do not enter into them here. Instead, we drop the AT assumption and pursue the line of investigation indicated by the query: What can be inferred about the boundedness of (4) by studying the relative disposition of the trajectories of the two "subsystems"

\[
\dot{x} = f_1(x) \tag{6}
\]

and

\[
\dot{x} = f_2(x) \tag{7}
\]

Classical stability theory concerns the extent to which study of the linear system (6) yields information about the qualitative behavior of solutions of (4) near \( x = 0 \). Analogous results might be expected from (7) about the behavior of (4) near a "point at infinity." Indeed, the AT case bears a certain resemblance to that "critical" case of plane stability theory in which (6) has a center and the higher order terms control the stability of \( x = 0 \) for (4). So it seems natural to look for "non-critical" cases of the boundedness problem in which the behavior of (7) is decisive for the boundedness of (4).

In Section 3 we examine several conjectures about necessary conditions on the system (7) for the boundedness or weak boundedness of (4). The examples constructed suggest the study of the special case in which the components of \( f_2(x) \) have a common quadratic factor (CQF); i.e., \( f_2(x) = (x, Cx)a \), where \( C \) is an \( n \times n \) symmetric constant matrix and \( a \) is a constant \( n \)-vector. For this CQF case, a boundedness theorem is established analogous to that for the AT case.
In Section 4 we turn to a detailed study of the boundedness of (4) in $\mathbb{R}^2$.
The necessary conditions of Section 3 are here found to limit the system (7)
to a few canonical forms and their affine equivalents. Two of these are of
the QGF type and one of the HE type. The remaining ones form a one-parameter
family whose trajectories are spiral arcs approaching rest points as $t \to +\infty$.
While the members of this family do not possess so simple an integral as in
the HE case, the two geometrical situations are quite analogous and the idea
of the HE case suffices to deal with the boundedness of (4) for this family,
too. As a consequence, we are able to find conditions on (6) both necessary
and sufficient for the boundedness of (4) when (7) has each of its admissible
forms; i.e., all bounded quadratic systems in $\mathbb{R}^2$ are characterized.

The remainder of Section 4 is devoted to a study of the structure of these
bounded quadratic systems in the plane. For this purpose, $\mathbb{R}^2$ is extended
to the Poincaré sphere in the well known manner, and the notion of a "reduced"
phase portrait is introduced. This device for obscuring ignorance of limit
cycle structure is based on the result of Tang that a closed orbit of any
plane quadratic system surrounds exactly one rest point. A "reduced" phase
portrait is one which associates a rest point with the $\omega$-limit set of a
trajectory by the ideograph $\to \omega$, which means that the trajectory approaches
either the rest point itself or a limit cycle which encloses the rest point
and is externally stable (and similarly for $\alpha$-limit set with $\to 0$ and external
instability). After study of the number, disposition, and types of rest
points occurring on the Poincaré sphere, we determine the reduced phase
portraits by examination of the separatrix structure.

The limit cycle structure to be found in the bounded plane systems can be
discussed by using results known for general quadratic systems and by applying
the theory of rotated vector fields; while this approach is quite promising, our results are not sufficiently complete to justify their exposition
here.

Section 5 digresses from the general theme of boundedness to present some
results on classification of general quadratic systems in $\mathbb{R}^2$ by the
character of the rest points on the Poincaré sphere.

Section 6 presents some special examples of "quadratic oscillators." Detailed discussion and proofs are omitted. A particularly simple plane system which shares the properties of (3) is described and related to a second order equation of Liénard type. A certain plane $AT$ system (used by E. Hopf in connection with a model of turbulence) is discussed by construction of a Ljapunov function and leads to a bounded quadratic system in $E^3$ having a stable limit cycle. Similar considerations lead also to a "pseudo-quadratic" system in $E^3$ having a stable torus as an invariant manifold, and this torus is seen to bifurcate from a limit cycle under parametric variation.
Section 2
ASYMPTOTICALLY TRANSVERSE SYSTEMS

This section presents a theorem which reduces the boundedness problem for a special class of quadratic systems on \( \mathbb{R}^n \) to the study of qualitative features of a (compact) dynamical system on \( S^{n-1} \).

1. A Criterion for Boundedness

The real quadratic autonomous system

\[ \dot{x} = f_0 + f_1(x) + f_2(x), \quad \text{on } \mathbb{R}^n, \quad (1) \]

where \( f_i \) denotes the terms of degree \( i \), will be called asymptotically transverse (AT) if the ternary form \( (x, f_2(x)) \) vanishes identically. We set \( f_1(x) = Ax \) where \( A \) is an \( n \times n \) matrix and \( x = \text{col}[x_1, \ldots, x_n] \), and denote the symmetric and skew-symmetric parts of \( A \) by \( \Lambda \) and \( \Lambda^s \)

\[ (A = \Lambda + \Lambda^s, \Lambda^T = \Lambda, \Lambda^T = -\Lambda). \]

The system

\[ \dot{x} = f_2(x) \quad (2) \]

will be called Homogeneous Transverse, or HT, and has the integral

\[ \|x\| = \sqrt{x_1^2 + \ldots + x_n^2} = \text{constant}. \]

Our objective here is to prove and to illustrate by some applications the

Theorem 1: Every solution of the AT system (1) is bounded if there exist positive \( T \) and \( \varepsilon \) such that

\[ \int_0^T (v, \dot{v}) dt < -\varepsilon \]

for every solution \( v = v(t) \) of the HT system (2) restricted to the unit sphere \( S^{n-1} : \{x| \|x\| = 1\} \).

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Proof: If \( x = ru \), \( r > 0 \), \( \|u\| = 1 \), system (1) is equivalent to

\[
\begin{align*}
\dot{r} &= (u, r) + r(u, -u) \\
\dot{u} &= \frac{r(u, f_o)u}{r} + Au - (u, Au) + rf_2(u),
\end{align*}
\]

or to

\[
\begin{align*}
\dot{r} &= \frac{u}{r} + (u, Au) \quad (3)_1 \\
\dot{u} &= \frac{r(u, f_o)u}{r^2} + Au - (u, Au)u + f_2(u) \quad (3)_2
\end{align*}
\]

on \( \mathbb{R}^n - \{0\} \), where \( \frac{d}{dt} = \frac{1}{r} \frac{d}{dt}r^m \). It therefore suffices to show that (3) defines a bounded system on \( S^m \). To do so, we restrict attention to an arc of a trajectory of (3) which lies outside a sufficiently large sphere, so that (3) may be regarded there as a "perturbation" of the system

\[
\begin{align*}
R' &= (v, \dot{v}) \quad (4)_1 \\
v' &= f_2(v), \quad \|v\| = 1 \quad (4)_2
\end{align*}
\]

The hypothesis of the theorem is equivalent to the existence of \( T, \varepsilon \) so that for every solution \( R, v \) of (4), \( R \) decreases by at least \( \varepsilon \) over any \( \tau \)-interval of length \( T \). We shall infer the existence of \( \rho > 0 \) such that, for any solution \( r, u \) of (3) with \( r(0) \geq \rho \), we have \( r(\tau) < r(0) \); i.e., no solution of (3) which starts at \( \tau = 0 \) from any sufficiently large sphere can remain outside that sphere for \( 0 < \tau < T \). Since (3)_1 limits \( r \) to linear growth with \( \tau \), such a solution is bounded.

Formally, let \( \varepsilon_1 > 0 \), \( \varepsilon_2 > 0 \) be chosen so that

\[
\left| \int_0^T \left[ (u_2, \dot{A}u_1) - (u_2, \dot{A}u_2) + h(\tau) \right] d\tau \right| < \varepsilon
\]
whenever \( u_1(\tau) \), \( u_2(\tau) \) are continuous paths in \( S^{n-1} \) such that
\[ ||u_1(\tau) - u_2(\tau)|| < \varepsilon_1 \] on \([0, T] \) and \( h(\tau) \) is a continuous real function such that \( |h(\tau)| < \varepsilon_2 \) on \([0, T] \).

Let \( g(x, \tau) \) denote a tangent vector field on \( S^{n-1} \) (continuous function on \( S^{n-1} \times [0, T] \rightarrow \mathbb{R}^n \) such that \( (x, g(x, \tau)) = 0 \)). Let \( v(\tau) \) solve \((h)\) with \( v(\tau, 0) = v_0 \) and \( v'(\tau) \) solve \( V = \frac{\partial}{\partial \tau} (v) + g(v, \tau) \) with the same initial value on \( S^{n-1} \): \( v(\tau, 0) = v_0(0) = v_0 \). By the continuous dependence theorem (uniform on compact \( S^{n-1} \)), there exists \( \varepsilon_3 > 0 \) such that \( ||v(\tau) - v(\tau')|| < \varepsilon_3 \) on \( S \times [0, T] \) provided that \( ||g(x, \tau)|| < \varepsilon_3 \).

Let \( k_0 = \max_{S^{n-1}} ||f_0 - (u_0, f_0) u|| = 2 \max_{S^{n-1}} ||(u, \tau)\| = 2\|f_0\| \) and \( k_1 = \max_{S^{n-1}} \|Au - (u, \tau)u\| = 2\|f_0\| \) and choose \( \delta \) so that both \( k_0/\delta < \varepsilon_2 \) and \( k_1/\delta < \varepsilon_3 \).

Now, let \( r(\tau), u(\tau) \) be a particular solution of \((3)\) such that \( r(0) \neq 0 \). A contradiction arises from the supposition that \( r(\tau) \neq r(0) \) for \( 0 < \tau < T \), as follows: Let \( R, v \) be the solution of \((h)\) such that \( R(0) = r(0), v(0) = u(0) \). Let \( h(\tau) = (u(\tau), f_0/v(\tau)) \)

\[ g(x, \tau) = \frac{f_0 - (x, f_0)x}{r^2(\tau)} + \frac{Ax - (x, Av)x}{R(\tau)} \]

it then follows from our supposition and the constructions above that

\[ \int_0^T [(u_1u) + h(\tau) - (v, Av)] d\tau \]

\[ = r(\tau) - r(0) - R(\tau) + R(0) = r(\tau) - R(\tau) < \varepsilon \]

But the hypothesis of the theorem means \( R(\tau) < R(0) - \varepsilon \). Hence \( r(\tau) < r(0) \), a contradiction.

2-3
2. Remarks:

1. If the hypothesis of the theorem is satisfied, the boundedness of (1) is not affected by arbitrary change of \( f_0 \) or \( \lambda \). Indeed, the conclusion follows also for the non-autonomous system obtained if \( f_0 \) and \( \lambda \) are bounded functions of \( t \).

2. It is clear from the proof that the boundedness of (1) is uniform in the sense of Section 1.

3. If the \(<\sim \varepsilon \) of the hypothesis is replaced by \( >\varepsilon \), a similar argument shows that every solution of (1) starting from a sufficiently large sphere is unbounded, but this is a weak converse. Various stronger converse statements are still open, and their discussion will be omitted here.

4. Of various generalizations of the theorem, the following is one of the simplest, and requires only minor modifications of the proof: Let (1) be replaced by a polynomial system of degree \( m \geq 2 \) : \( \dot{x} = f_0 + Ax + f_2(x) + \ldots + f_m(x) \) for which \( (x, f_k(x)) \equiv 0 \) \( k = 2, 3, \ldots, m \), and let (2) be replaced by \( \dot{x} = f_m(x) \). Then the theorem as stated holds.

5. If all \( \alpha \)-limit sets of trajectories of (2) on \( S^{n-1} \) lie in that open subset \( S_c \subset S^{n-1} \) on which \( (x, Ax) < 0 \), then so do all \( \alpha \)-limit sets (since \( \alpha \)- and \( \omega \)-limit sets form disjoint antipodal pairs); hence \( \eta > 0 \) exists so that all points \( x \in S^{n-1} \) where \( (x, Ax) \geq -\eta \) are wandering points. It follows that \( \int_0^t (v, \dot{x}) dt \rightarrow -\infty \) as \( t \rightarrow \infty \) and this limit is approached uniformly for \( v_0 \in S \). The hypothesis of the theorem is a fortiori satisfied in this case. It may be satisfied in other cases as well, however; the third example below illustrates the possibility of periodic solutions of (2) which intersect both \( S_c \) and \( S_\varepsilon = \{ x : |x| = 1, (x, \lambda x) > 0 \} \).

6. If \( C_1, C_2, \ldots, C_n \) are \( n \times n \) constant skew-symmetric matrices, \( \dot{x} = \sum_{k=1}^n x_k C_k x \) is a skew-symmetric matrix such \( \dot{x} = B(x)x \) is HT, as
mentioned in Section 1. While this notational representation for \( f_2(x) \) makes the orthogonality condition obvious, its lack of uniqueness should be noted; given \( f_2(x) \), there are many different \( B(x) \)--skew symmetric and otherwise--such that \( f_2(x) = B(x)x \). Perhaps curiously at first sight, there is a unique symmetric such \( B(x) \) (proof omitted).

7. In Section 1 and generally in the remainder of this report, we discuss quadratic systems with a rest point at the origin. It may be asked what generality is gained by including constant terms on the right hand side of (1). We have not yet resolved this problem; i.e., we do not know whether a bounded quadratic system must possess at least one rest point. If one exists, translation of the origin to it eliminates the constant terms. (Bhatia and Szego [1] have recently published the conjecture that any (positively) bounded dynamical system on \( \mathbb{R}^n \) has a rest point.)

8. An affine invariant version of the theorem is easily formulated and applies to systems for which the associated homogeneous quadratic system has central ellipsoids as invariant hypersurfaces.

3. Examples

1. On \( \mathbb{R}^2 \) it follows from the Poincaré-Bendixson theory that a bounded system has a rest point and we may assume the origin located at one. As will be seen in Section 4, a similarity transformation suffices to put any plane AT system in the canonical form

\[
\begin{align*}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + x_2^2 \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 - x_1x_2
\end{align*}
\]

The phase-portrait of the HT system shows the \( x_1 \)-axis as a line of rest points and all other trajectories are semi-circular arcs approaching rest points on the positive \( x_1 \)-axis (cf. Fig. 4, Section 4). The situation of
Remark 5 obtains if the $x_1$-axis belongs to the contraction cone $(x, \hat{A}x) < 0$; the condition for this is clearly that $a_{11} < 0$. If $a_{11} > 0$, unboundedness prevails (Remark 3). If $a_{11} = 0$, the theorem does not apply; it will be seen in Section 4 that if $a_{11} = 0$, the system is bounded if and only if either $a_{21} = 0$ (a degenerate case) or $a_{22} = -a_{21}$ and $a_{22} < 0$. Thus, the theorem is reasonably sharp in that it catches all the non-degenerate bounded cases in which the expansion cone is non-vacuous.

2. In $\mathbb{R}^3$ homogeneous systems can be studied by using polar coordinates and $\tau$-time as in (3). The resulting decoupling of the radial variation from the flow on $S^2$ permits a kind of extension of the Poincaré-Bendixson theory to $\mathbb{R}^3$; this idea was worked out by Coleman [2]. In the quadratic HT case, $f_2(x)$ admits a special decomposition which often can be used to "visualize" the nature of the flow on $S^2$; we write $f_2(x) = x \otimes Mx$ where $M$ is a constant matrix, $x$ is a column vector and $\otimes$ denotes the ordinary vector product for 3-vectors. The decomposition $f_2 = f_2^S + f_2^A$ is obtained by splitting $M$ into symmetric and skew symmetric parts, $M = \hat{N} + \tilde{N}$. If one of these vanishes, the situation is particularly simple: Define $\mu > 0$ and a unit vector $m$ by

$$\tilde{N} = \mu \begin{bmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{bmatrix}$$

If $\tilde{N} = 0$, $\tilde{N} \neq 0$, $\dot{x} = x \otimes \tilde{N}x = \mu x \otimes (m \otimes x) = \mu (\|x\|^2 m - (x,m)x)$ defines a flow along great circles from a "source", at the point $-m$ of $S^2$ to a "sink" at the point $m$. (This is a gradient flow on $S^2$ in the sense that $\mu (m - (x,m)x)$ is the tangential component on $S^2$ of grad $(x,\mu m)$.)

If, on the other hand, $\hat{N} = 0$, $\hat{N} \neq 0$ the flow defined by $\dot{x} = x \otimes \hat{N}x$ has the quadratic form $(x,\hat{N}x)$ as an integral and the trajectories lie along the curves in which $S^2$ intersects the quadric surfaces $(x,\hat{N}x) = \text{constant}$. (The Euler equations for the force-free motion of
a rigid body with one fixed point are a classical example, in which the
integrals \((x, x)\) and \((\dot{x}, \dot{x})\) correspond to energy and angular momentum.

To construct an example which combines these two kinds of flow in a per-
spicuous way, let \(\mathbf{N} = \text{diag} \begin{bmatrix} \omega, 0, 0 \end{bmatrix} \), \(\omega > 0\), and \(\mathbf{m} = \text{col} \begin{bmatrix} 0, 0, 1 \end{bmatrix} \),
\(\mu = 1\). Then \(\mathbf{F}_2(x) = \text{col} \left[ -\omega y, \omega x, 0 \right]\) and \(\mathbf{F}_3(x) = \text{col} \left[ -x_2 x_3, x_2^2 + x_3^2 \right]\). The \(x = F_2(x)\) flow is everywhere "northward" from
the south pole at \(-m\) to the north pole at \(m\). The \(x = F_3(x)\) flow is
directed "eastward" around latitude circles in the northern hemisphere and
westward in the southern hemisphere. The combined \(x = F_2(x) + F_3(x)\) flow
is clearly one which spirals northwesterly in the southern hemisphere and
northeasternly in the northern hemisphere. (Incidentally, any plane through
the origin clearly intersects every trajectory except possibly the rest
points, disproving a conjecture—mentioned in our Annual Status Report to
APOS— that any full trajectory of a homogeneous quadratic system can be
separated from its negative by a hyperplane.) Using this \(F_2(x)\), we see
from Remark 3 that \(x = A x + F_2(x)\) is bounded in \(E^3\) if \(a_{12} < 0\).

It may be noticed in connection with this example that the condition
\(a_{23} < 0\) can be satisfied by matrices \(A\) having all eigenvalues positive.
Every non-trivial solution of the linear system \(\dot{x} = A x\) will then be
unbounded, yet the addition of quadratic terms having no radial component
results in a bounded system. By a change of scale, the same holds for the
system \(\dot{x} = A x + \epsilon F_2(x)\) for any \(\epsilon \neq 0\). This "paradox" reveals the
subtle aspect of Theorem 1 as compared to boundedness achieved by the
qualitative notion of "repulsion from infinity" discussed in Section 1.

3. In \(E^3\), let \(f_1(x) = A x\) with \(A = \mathbf{N} = \text{diag} \begin{bmatrix} -1, -1, 0 \end{bmatrix}\) and \(f_2(x) =
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.

\[ M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \]
Thus,

$$\dot{x}_1 = -x_1 - x_2 x_3 - x_3^2$$
$$\dot{x}_2 = -x_2 + x_1 x_3$$
$$\dot{x}_3 = \alpha x_2 + x_1 x_3$$

By the geometric method of Example 2, we see that the trajectories of $k = f_2(x)$ on $S^2$ are:

- Rest points at $\pm \frac{1}{2}(0, \sqrt{E}, \sqrt{E})$
- Rest points at each point of the "equator" $x_3 = 0$
- The closed orbits (circles) in which $S^2$ is intersected by the planes $|x_2 - x_3| = \text{const} > 1$
- The separatrix cycles $S^2 \cap \{x \mid |x_2 - x_3| = 1\} \cap \{x \mid x_3 \neq 0\}$
- The circular arcs $S^2 \cap \{x \mid |x_2 - x_3| = \text{const} < 1\} \cap \{x \mid x_3 \neq 0\}$

If $0 < \alpha < 1$, the expansion cone, $-x_1^2 - x_2 + \alpha x_3^2 > 0$, intersects $S^2$ in spherical caps centered at the poles $\pm(0, 0, 1)$. These caps are therefore traversed by both periodic and non-periodic orbits of the HT flow. It is obvious for sufficiently small $\alpha$ (and can be verified for all $\alpha \in (0, 1)$) that all orbits spend "most of the time" in the contraction cone, and that an $s, T$ satisfying the hypothesis of Theorem 1 will exist. As $\alpha \to 1$, the spherical caps spread out from the poles; when $\alpha = 1$, the rest points at $\pm \frac{1}{2}(0, \sqrt{E}, \sqrt{E})$ are themselves trajectories for which $(x, Ax) = 0$, so that the hypothesis of the theorem cannot be satisfied. (That boundedness fails in this case is already suggested by the observa-
tion that, aside from the unstable origin, the full system $\dot{x} = Ax + f_2(x)$ has only the two rest points \((-\alpha, \frac{\alpha}{1-\alpha})^{\frac{1}{2}}, \frac{\alpha}{1-\alpha}^{\frac{1}{2}}\), and these approach infinity as $\alpha \to 1$.

4. The FT system $\dot{x} = f_2(x)$ where

\[
\begin{align*}
\dot{x}_1 &= x_2^2 \\
\dot{x}_2 &= -x_1 x_2 + x_3^2 \\
\dot{x}_3 &= -x_2 x_3 + x_4^2 \\
&\vdots \\
\dot{x}_{n-1} &= -x_{n-2} x_{n-1} + x_n^2 \\
\dot{x}_n &= -x_{n-1} x_n
\end{align*}
\]

displays "local coupling", in the sense that the $k$th component of $f_2(x)$ depends on none of the $x_j$ with $|k-j| > 1$. It is seen by recursive inspection that the only rest points on $S^{n-1}$ are $P_\pm = \pm(1,0,0,\ldots,0)$. That every other point on $S^{n-1}$ lies in a trajectory approaching $P_+$ can be seen from the following argument. Let $\Omega$ denote the (non-empty, compact, invariant) $\omega$-limit set of the trajectory from an arbitrary initial point of $S^{n-1}$. From the first equation, $x_1(t)$ is non-decreasing along this trajectory so $\lim_{t \to 1} x_1(t) = \alpha$ exists and $\Omega$ therefore lies in the intersection of $S^{n-1}$ with the hyperplane $x_1 = \text{constant} = \alpha$. Now consider the trajectory issuing from a point $x \in \Omega$. On this trajectory $x_1(t) = \alpha$, $\dot{x}_1(t) = 0$, and we see from the first equation that $x_2(t) = 0$. Hence $\dot{x}_3(t) = 0$, so that $x_3(t) = 0$ from the second equation, $x_4(t) = 0$ from the third, and so on to $x_n(t) = 0$. Hence $x_1^2(t) = 1$ and $\alpha = -1$ or +1 according as the initial trajectory is or is not the point $P_-$ itself. Again by Remark 5, the FT system $\dot{x} = Ax + f_2(x)$ is bounded if $a_{11} < 0$.
(If we add the term $-x_1 x_n$ to the right side of the first equation of the system above and the term $+x_1^2$ to the last equation, the system acquires a cyclic structure in the sense that $x_k = F(x_{k-1}, x_k, x_{k+1})$ for $k = 1, 2, \ldots, n$, where $F$ is independent of $k$ and we identify $x_0 = x_n$, $x_{n+1} = x_1$; we have studied various examples of this general type arising as approximations to certain partial differential equations, and hope to elaborate these studies in future work, but we forego an exposition here.)

Section 2 References


Section 3

Behavior Near Infinity of Quadratic Systems on \( \mathbb{R}^n \)

This section contains observations about boundedness for

\[ \dot{x} = f_1(x) + f_2(x), \quad f_1(x) = Ax \tag{1} \]

in case the associated homogeneous quadratic system

\[ \dot{x} = f_2(x) \tag{2} \]

is not known to be affinely equivalent to the H.T. case of Section 2.

1. Sufficient Conditions for Unboundedness

We note first some implications of the homogeneity and parity of \( f_2(x) \).

In polar form \( (x = ru, \ r \geq 0, \ ||u|| = 1) \) the trajectories of (2) coincide for \( r > 0 \) with those of

\[ \begin{align*}
\dot{r} &= r(u, f_2(u)) \tag{3.1} \\
\dot{u} &= f_2(u) - (u, f_2(u))u \tag{3.2}
\end{align*} \]

where \( \dot{u} = \frac{d}{dt} u = \frac{1}{r} \frac{d}{dt} \). That is, the solution of (2) such that \( x(0) = x_0 = r_0 u_0 \) is given by \( x(t) = r(\tau(t)) u(\tau(t)) \), where

\[ r(\tau(t)) = \int_0^\tau (u, f_2(u)) \, d\tau \]

in which \( u = u(\tau) \) is the solution of the autonomous system (3.2) on \( \mathbb{S}^{n-1} \) such that \( u(0) = u_0 \), and \( \tau(t) \) is determined by

\[ t = \int_0^{\tau(t)} \frac{dt}{r(\tau)} = \int_0^{\tau(t)} e^{\int_0^\tau (u, f_2(u)) \, d\tau} \, ds . \]
If $u = u_0$ is a rest point of $(3)_2$ such that $(u_0, f_2(u_0)) > 0$, we call the corresponding solution of $(2)$ a ray solution, since its trajectory is the ray in the direction of $u_0$. If $(u_0, f_2(u_0)) < 0$, then there is a ray solution in the direction $-u_0$. On a ray trajectory $r_2(x) = r_2^2 f_2(u_0) = \lambda r^2 u_0$, $\lambda > 0$, and $(2)$ reduces to the one-dimensional equation $\tau = \lambda r^2$; hence a ray solution is one of finite escape time.

If $(u_0, f_2(u_0)) = 0$, the line $su_0$, $-\infty < s < +\infty$, is a line of rest points for $(2)$. Such a line exists iff the origin is not an isolated rest point for $(2)$.

If a trajectory $x(t)$ of $(2)$ intersects a ray at two distinct points $x(t_1)$ and $x(t_2) = \alpha x(t_1)$ where $\alpha > 1$ and $t_2 > t_1$, then the trajectory is unbounded, since the corresponding trajectory of $(3)_2$ must be either a rest point or a periodic orbit of period $T > 0$. In the former case, we have a ray solution; in the latter case, $\int_0^T (u, f_2(u)) dt > 0$, and we see from $(3)_1$ that $x(\tau') - e$ as $e \to -\infty$. If $t_1 > t_2$, on the other hand, $-x(-t)$ is unbounded.

A trajectory of $(2)$ which contains a point $x_0 \neq 0$ cannot intersect the ray containing $-x_0$. If it did, there would be a trajectory $u(\tau)$ of $(3)_2$ containing antipodal points of $S^{n-1}$. But if $u(\tau_1) = -u(\tau_2)$, then $u(\tau + \tau_1)$ and $-u(\tau + \tau_2)$ are solutions of $(3)_2$ with the same initial value, so $u(\tau + \tau_1) = -u(\tau + \tau_2)$, and hence $\tau = \frac{3}{2}(\tau_2 - \tau_1)$ gives $u(\frac{3}{2}(\tau_2 - \tau_1)) = -u(\frac{3}{2}(\tau_2 + \tau_1))$, a contradiction.

The following theorem is due to Markus [1] in case $n$ is odd or $n = 2$.

**Theorem 1**: A ray solution of $(2)$ exists if the origin is an isolated rest point.

**Proof**: It suffices to show that $(3)_2$ must have a rest point on $S^{n-1}$. If $n - 1$ is even, this is an immediate consequence of the well known...
theorem of Brouwer on the non-existence of continuous direction fields on even-dimensional spheres. If \( n-1 \) is odd, however, it depends on the even parity of the right side of (3)\(_2\). Let \( G(u) \) denote this right side; if \( 0(u) \neq 0 \) on \( S^{n-1} \) then \( H(u) = G(u)/\|G(u)\| \) defines a map \( S^{n-1} \to S^{n-1} \) which collapses antipodal points and hence has even topological degree if \( n-1 \) is odd (by Theorem 2 of [2]). On the other hand, the orthogonality of \( u \) and \( H(u) \) implies that \( H \) sends no point to its antipode, and therefore (by Corollary 4(a) of [2]) \( H \) is homotopic to the identity and \( \deg H = 1 \), a contradiction.

**Corollary 1:** In the PT case of Section 2, the system (2) has at least one line of rest points, therefore at least two rest points on \( S^{n-1} \).

**Theorem 2:** If (2) has a ray solution, the system (1) has an unbounded solution of finite escape time.

**Proof (sketched):** The system

\[
r' = (u, f_1(u)) + r(u, f_2(u))
\]

\[
u' = \frac{3}{2}(f_1(u) - (u, f_1(u))u) + f_2(u) - (u, f_2(u))u
\]

arises from (1) as (3) did from (2). The assumption of a ray solution for (2) means that \( u_0 \) exists so \( f_2(u_0) - (u_0, f_2(u_0))u_0 = 0 \) and \( (u_0, f_2(u_0)) = \lambda > 0 \). The substitution \( \sigma = \frac{1}{r} \) in (4) yields

\[
\sigma' = -\sigma(u, f_2(u)) - \sigma(u, f_1(u))
\]

\[
u' = \sigma(f_1(u) - (u, f_1(u))u) + f_2(u) - (u, f_2(u))u
\]

To consider this system in a neighborhood of the rest point \( \sigma = 0 \), \( u = u_0 \), we set \( y_1 = \sigma \) and let \( y_2, \ldots, y_n \) be local analytic coordinates in a neighborhood of \( u_0 \) on \( S^{n-1} \).
Then (5) is equivalent near \( y = 0 \) to

\[ y = Py + \mathcal{P}_2(y) \tag{6} \]

where \( \mathcal{P}_2(y) \) denotes a power series starting with terms of degree 2 and convergent in a neighborhood of \( y = 0 \). From (5), we see that the first row of \( P \) is \([-\lambda, 0, 0, ..., 0]\), so that \(-\lambda\) is a negative eigenvalue of \( P \).

A classical theorem of Lyapunov (Theorem 1.141, p. 235 of [3]) associates this eigenvalue with at least one trajectory of (6) which approaches the origin. The constructive method of proof of this theorem reveals in the present application that there exists such a trajectory on which \( Y_2 > 0 \)

(note that \( Y_2(\tau_k) = 0 \) implies \( Y_2(\tau) \equiv 0 \) from (5)). Inverting the transformations, it follows that (4) has a solution \( x(\tau) \), \( u(\tau) \) such that \( u(\tau) \rightarrow u_0^2 \), \( r(\tau) \rightarrow \infty \) as \( \tau \rightarrow \infty \). It can be inferred from (4) that \( r(\tau) \) grows exponentially with \( \tau \), and the escape time for the corresponding solution of (1) is therefore finite.

We state the following stronger result as:

**Conjecture 1:** If (2) has an unbounded solution of finite escape time, so does (1).

Weaker versions are:

**Conjecture 2:** If (2) has an unbounded solution of finite escape time, then (1) has an unbounded solution.

**Conjecture 3:** If \( u_0 \) is an rest point and \( u(\tau) \) a solution of (3) such that \( u(\tau) \rightarrow u_0 \) and

\[ \int_0^\infty \exp\left(-\int_0^\tau (u, f(u))d\tau\right)ds < \infty, \]

then (1) has an unbounded solution.
2. Examples

1. In $E^2$ with $f_2(x) = [x_1^2, x_2, x_1^2 + x_1^2]$ the system (2) has no ray solution but does have solutions of finite escape time (in $x_1 > 0$). The hypotheses of Conjecture 3 are satisfied (for $u_0 = [0,1]$) and so is the conclusion, as will be proved in Section 4.

2. In $E^2$, the example $f_2(x) = [x_1, x_2, 0]$, $f_1(x) = [0, x_1]$ shows that (1) may have solutions of finite escape time even if (2) has no such solutions.

3. With the same $f_2(x)$ as in Example 2 and $f_1(x) = [0, -x_2]$ it is seen that the system (1) may be bounded even though (2) is not.

4. In $E^3$, the example of system (1) with $f_2(x) = [0, 0, x_1 x_2]$, $f_1(x) = [x_2, -x_1, 0]$ illustrates the same possibility as Example 3, but has only rest points and periodic solutions, while the system (1) of Example 3 has every solution approaching a rest point.

5. In $E^3$, the system (2) with $f_2(x) = [x_1 x_2, x_2 x_3, -x_1 x_3 + x_2 x_3, x_1^2 + x_2^2]$ has solutions of finite escape time, but the corresponding (3) has no rest point $u_0$ satisfying the hypothesis of Conjecture 3. In fact, if $r(\tau)$, $u(\tau)$ is an unbounded solution of (3), then $u(\tau)$ has a limit cycle of (3) as its $\omega$-limit set. (These statements become apparent if cylindrical coordinates $r = \sqrt{x_1^2 + x_2^2}$, $\theta = \tan^{-1} x_2/x_1$, and $z = x_3$ are employed.) Thus, Conjecture 2 is actually stronger than Conjecture 3.

6. That not all bounded systems (3) are reducible to the HT case will be seen from the example $f_2(x) = [x_1^2, -x_1 x_2 + x_2^2]$, $(0 < c < 2)$, which will be studied in Section 3.

7. System (2) may be called a "square" system if the components of $f_2(x)$ are not all zero and contain only squares of the $x_1$. A system (2) is equivalent to a square system under linear transformation if its component...
Quadratic forms have matrices which are mutually commutative and not all zero. The example \( f_2(x) = [x_2^2, 0] \) shows that a square system \((2)\) need not have a ray solution. But we have not succeeded in proving or disproving:

Conjecture 4: A square system is unbounded.

8. The example

\[
\dot{x} = f_2(x) = [x_1^2, (x_2^2 + \cdots + x_n^2), 2x_1x_2, 2x_1x_3, \ldots, 2x_1x_n]
\]

shows that a system \((2)\) may have exactly one ray solution and every other solution bounded. In cylindrical coordinates \( x = r \) and \( y = (x_2^2 + \cdots + x_n^2)^{1/2} \), this system gives

\[
\dot{x} = x^2 - y^2
\]

\[
\dot{y} = 2xy
\]

which has the positive \( x \)-axis as a ray solution; all other trajectories in \( y > 0 \) are the circles tangent to the \( x \)-axis at the origin.

3. CGF Systems

The Examples 3 and 4 above show that \((1)\) may be bounded, though \((2)\) is not. We consider more generally here the class of systems \((1)\) for which the components of \( f_2(x) \) have a common quadratic factor (CGF); i.e., for which

\[
f_2(x) = (x, Cx)a, \quad C = 0, \quad \|a\| = 1
\]

(7)

where \( C \) is a symmetric \( n \times n \) matrix and \( a \) is a unit \( n \)-vector. In this case, every trajectory of \((2)\) falls in one of the categories.
a. Rest points, \[ x = (C; x, C) = 0 \]

b. Half-lines parallel to \( a \) and approaching a rest point as \( t = 0 \) and \( e \) as \( t \to -\infty \) (or \( e = \infty \) and a r.p. as \( t \to -\infty \)).

c. Full lines parallel to \( a \) and approaching \( e \) as both \( t \to -\infty \).

d. Line segments parallel to \( a \) and approaching rest points as both \( t \to -\infty \).

In particular, the ray from the origin in the direction of \( a \) is a ray solution unless it consists of rest points. Hence, by Theorem 2, we have:

**Theorem 3:** In the CQP case, \((7)\), a necessary condition for the boundedness of \((1)\) is

\[
(a, Ca) = 0 \quad .
\]  

(9)

A line in \( \mathbb{R}^n \) parallel to \( a \) and containing \( b \) can be parametrized by \( x = b + sa \), \(-\infty < s < +\infty\). Since \((b + sa, C(b + sa)) = (b,Cb) + 2s(b, Ca) + s^2(a, Ca)\), condition (9) implies that the line contains exactly one rest point if \((b, Ca) \neq 0\) and \((b, Cb) \neq 0\), no rest point if \((b, Ca) = 0\), \((b, Cb) \neq 0\), and nothing but rest points if \((b, Ca) = 0\), \((b, Cb) = 0\). (No trajectories of type (8)\(\dot{A}\).) On the line, \( \dot{A} = (b, Cb) + 2s(b, Ca) \), a linear equation, hence:

**Corollary 2:** In the CQP case, \((7)\), condition \((9)\) is necessary and sufficient for the weak boundedness (cf. Section 1) of system \((2)\).

To find sufficient conditions for \((1)\) to be bounded, assuming \((7)\) and \((9)\), we put \( x = ya + a \), where \( y = (x, a) \), \( z = x - (x, a)a \), and therefore \((z, a) = 0\). System \((1)\) is then equivalent to

\[
\begin{align*}
\dot{y} &= (a, Ay) + 2y(a, Ca) + (z, Cz) \\
\dot{z} &= Az - (a, Ay) + y(Aa - a, Aa)a
\end{align*}
\]  

(10)_1

(10)_2
Equation (10) governs the flow in a-direction in \( \mathbb{R}^n \), while (10)_2 governs the projection of the flow on the \( n-1 \) dimensional hyperplane orthogonal to \( a \). The situation in which Equation (10)_2 "uncouples" leads to:

**Theorem 4:** In the CSP case, (7), the system (1) is bounded if, in addition to (5),

a. \( a \) is an eigenvector of \( A, Aa = \lambda a \).

b. The linear system

\[
\dot{x} = Ax - (a, Ax)a = (I - aa^T)Ax
\]

is bounded.

c. For any solution \( z = z(t) \) of Equation (11), the (linear, non-homogeneous, time dependent) equation

\[
\dot{y} = [\lambda z + (a, Cz)]y + (a, Ax) + (z, Cz)
\]

is bounded.

The boundedness problem is therefore reduced to one for linear equations in case \( a \) is an eigenvector of \( A \). Example 3 above illustrates the situation in which the origin is asymptotically exponentially stable for (11) and \( \lambda < 0 \), so that (12) is bounded. Example 4 is a case where (11) has periodic solutions and \( \lambda \leq 0 \). It will be seen in Section 4 that all bounded CSP systems in \( \mathbb{R}^2 \) are covered by Theorem 4.

**Section 3 References**


Section 4

BOUNDED QUADRATIC SYSTEMS IN $\mathbb{R}^2$

This section contains a study of those two-dimensional systems with quadratic polynomial right hand sides which have all of their trajectories bounded for $t \geq 0$. Such systems will be referred to as bounded quadratic systems. Any such system will have a rest point in the plane and by translating the origin to such a rest point it will have the form

$$\dot{x} = Ax + f_2(x), \quad x \in \mathbb{R}^2$$  \hspace{1cm} (1)

where $A$ is a matrix $[a_{ij}]$, $i, j = 1, 2$; and where the components of $f_2(x)$ are homogeneous quadratic polynomials in $x = (x_1, x_2)$. It will be assumed that $f_2(x) \neq 0$ since linear systems can be integrated in terms of elementary functions.

The survey paper of Coppel [1] contains most of the important results for quadratic systems in the plane. At the end of his paper, Coppel states that what remains to be done for quadratic systems is to determine all possible phase portraits and, ideally, to characterize them by means of algebraic inequalities on the coefficients.

This section first of all establishes necessary and sufficient conditions for a two-dimensional quadratic system to have all of its trajectories bounded for $t \geq 0$ and then determines all possible phase portraits for such bounded quadratic systems in the plane. In determining all possible phase portraits for bounded quadratic systems, some characterization by means of algebraic inequalities on the coefficients of system (1)---or (1) under a suitable linear transformation of coordinates---is also accomplished. In order to complete this type of algebraic characterization, there remains the problem of determining algebraic inequalities on the
coefficients which decide the number and stability properties of limit cycles around each isolated rest point. This remains the outstanding unsolved problem for bounded quadratic systems in the plane.

1. Classification of Bounded Quadratic Systems

Markus has classified the homogeneous quadratic systems in the plane up to affine transformations by classifying the related real linear algebras; cf. Theorems 6-8 of [2]. The homogeneous quadratic systems corresponding to the algebras in Theorems 7 and 8 and to those in Theorem 6, cases 3), 4), 6), 8) and case 10) with \( k = -1/3 \) all have ray solutions; i.e., the related algebra has an idempotent element. Now if the homogeneous quadratic equation

\[ \dot{x} = f_2(x) \]

has a ray solution then the quadratic system (1) will have an unbounded solution; cf. Section 3. Thus, the system (1) has an unbounded solution unless the quadratic part of the right hand side, \( f_2(x) \), reduces to one of the following forms (corresponding to the algebras of Theorem 6, cases 2), 5), 7), 9) and case 10) for \( k < -1/3 \), the last two cases corresponding to the last form below) under a suitable affine transformation:

A) \[
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\]

B) \[
\begin{pmatrix}
x_2^2 \\
0
\end{pmatrix}
\]

C) \[
\begin{pmatrix}
x_1x_2 + x_2^2 \\
x_2^2
\end{pmatrix}
\]

D) \[
\begin{pmatrix}
x_2^2 \\
-x_1x_2 + cx_2^2
\end{pmatrix}, \quad |c| < 2
\]
Case 1) of Theorem 6 corresponds to the linear system (1) with $f_2(x) = 0$; the global behavior for such systems is well known, cf., e.g., [3], and they will not be considered here. The "phase-portraits" for the homogeneous quadratic systems corresponding to A) - D) are given in Figs. 1-4 respectively. Note that each homogeneous quadratic system corresponding to A) - C) has an unbounded trajectory.

![Fig. 1](image1.png) ![Fig. 2](image2.png) ![Fig. 3](image3.png)

![Fig. 4](image4.png)

First let us show that the system (1) with $f_2(x)$ given in C) has an unbounded trajectory (even though the corresponding homogeneous quadratic system has no ray solution).
Lemma 1: The system

\[ \dot{x} = Ax + (x_1^2, x_2^2) \quad x(0) = x_0 \quad (2) \]

has an unbounded trajectory \( (t \to \infty) \) for some \( x_0 \in \mathbb{R}^2 \).

Proof: Let \( D_{a \beta} = \{(x_1, x_2) : x_1 > \alpha, x_2 > \beta \} \) where \( \alpha, \beta > 0 \). Since the only case when the rest points of (2) are not finite in number is when \( x_2 = -a_{12} \) is a line of rest points (i.e., when \( a_{21} = 0 \) and either \( a_{11} = a_{22} = 0 \) or \( a_{11} = a_{22} = a_{21} \)), it follows that for any given matrix \( A \), \( \alpha \) and \( \beta \) can be chosen sufficiently large that \( D_{a \beta} \) contains no rest points of (2). On \( x_1 = \alpha \)

\[ \dot{x}_1 = x_2(x_2 + a_{12}) + a_{11} \]

which is positive for all \( \alpha > 0 \) and \( x_2 \geq \beta \) provided \( \beta \) is sufficiently large. On \( x_2 = \beta \)

\[ \dot{x}_2 = a_{21}x_1 + \beta(a_{22}) \]

which for \( a_{21} > 0 \) is positive for all \( \beta > 0 \) and \( x_1 \geq \alpha \) provided \( \alpha \) is sufficiently large. And for \( a_{21} = 0 \) the above expression for \( \dot{x}_2 \) is positive for \( \beta > 0 \) sufficiently large. Thus, for \( a_{21} > 0 \), the system (2) has an unbounded trajectory \( (t \to \infty) \) for some \( x_0 \in D_{a \beta} \). This follows, for example, from Hartman's Theorem 1.1, p. 202 in [4]. For \( a_{21} < 0 \) consider the curve \( V(x_1, x_2) = x_1 - kx_2^2 = 0 \) for \( x_2 > 0 \). The flow defined by (2) is upward across \( V(x_1, x_2) = 0 \) for \( 0 < k < 1/(2|a_{21}|) \) and \( x_2 \) sufficiently large. This follows since from (2) and the definition of \( V(x_1, x_2) \)

\[ \frac{dV}{dt} = \dot{x}_1 - 2kx_2 \dot{x}_2 = -k(1 + 2a_{21}k)x_2^3 + \dot{x}_2^2 > 0 \]

on \( V(x_1, x_2) = 0 \) for \( 0 < k < 1/(2|a_{21}|) \) and \( x_2 \) sufficiently large.
Let
\[ D_{\alpha \kappa} = \{(x_1, x_2); x_1 \geq \alpha, x_2 \geq \frac{1}{\kappa} x_1 \} \]

Then for \( a_{21} < 0, 0 < \kappa < 1/(2|a_{21}|) \) and \( \alpha > 0 \) sufficiently large
\( D_{\alpha \kappa} \) contains an unbounded trajectory of (2) as \( t \to \infty \). This completes
the proof of Lemma 1.

We next consider Equation (1) with the quadratic part of \( f_2(x) \) given in A.

**Lemma 2:** The system
\[ \dot{x} = Ax + \begin{pmatrix} \vdots \\ x_1 x_2 \end{pmatrix}, \quad x(0) \in \mathbb{R}^2 \]  
(3)

has all of its trajectories bounded (for \( t \geq 0 \)) if and
only if \( a_{12} = 0, a_{11} < 0 \) and \( a_{22} < 0 \).

**Proof:** The critical points at infinity for (4) can be represented by the
pairs of diametrically opposite points \( P_1(1,0,0) \) and \( P_2(0,1,0) \) on
the equator of the sphere \( S^2 \) in \( E^3 \); cf. Lefschetz [3, p. 201].

![Diagram of the Poincaré Sphere and Local Coordinates](image)

**Fig. 5:** The Poincaré Sphere and Local Coordinates

4.5

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In terms of local coordinates \((x_1, x_2)\) at the point \(P_0\), cf. Fig. 5, Equation (3) has the form

\[
\begin{align*}
\dot{x}_1 &= a_{12} x_2 + (a_{11} - a_{22}) x_1 x_2 - x_1^2 - a_{21} x_1^2 x_2 \\
\dot{x}_2 &= -x_1 x_2 - a_{21} x_1^2 - a_{11} x_2^2
\end{align*}
\]

For \(a_{12} \neq 0\) this system has an elliptic sector at \((0,0)\). This follows, for example, from Theorem 66 on p. 297 of Andronov and Leontovich [5]. This implies the existence of unbounded trajectories for (3) as \(t \to \infty\) in case \(a_{12} \neq 0\). For \(a_{12} = 0\), Equation (3) is integrable. We obtain for \(a_{12} = 0\), \(a_{11} \neq 0\)

\[
\begin{align*}
x_1(t) &= x_1(0) e^{a_{11} t} \\
x_2(t) &= x_2(0) \exp\left[\frac{\int_0^t x_1(0) e^{a_{11} \tau} \, d\tau}{a_{11}}\right]
\end{align*}
\]

If \(a_{11} > 0\) and \(x_1(0) \neq 0\) then \(|x_1(t)| \to \infty\) as \(t \to \infty\). If \(a_{11} < 0\), \(x_2(t) \to 0\) as \(t \to \infty\) and it follows in this case that \(x_2(t) \to 0\) as \(t \to -\infty\) if \(a_{22} < 0\); \(|x_2(t)| \to \infty\) if \(a_{22} > 0\) and \(x_2(0) \neq 0\); and \(x_2(t) - x_2(0) - x_1(0) a_{21}/a_{11}\) as \(t \to \infty\) for \(a_{22} = 0\). The remaining case when \(a_{12} = a_{11} = 0\) is integrable and obviously has unbounded solutions (as \(t \to \infty\)) for \(x_1(0)\) sufficiently large. This completes the proof of Lemma 2.

Let us next consider the system (1) with \(f_0(x)\) given in B).
Lemma 3: The system

\[ \dot{x} = Ax + \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}, \quad x_2(0) \in \mathbb{R}^2 \]  \hspace{1cm} (5)

has all of its trajectories bounded (for \( t > 0 \)) if and only if \( a_{21} = 0, a_{11} < 0, a_{22} < 0 \) and \( a_{11} + a_{22} < 0 \).

Proof: The system (5) has only one critical point at infinity represented by the pair of diametrically opposite points \( P_1(1,0,0) \). In terms of local coordinates \((x_2, x_3)\) at \( P_1 \), cf. Fig. 5, Equation (5) has the form

\[ \begin{align*}
\dot{x}_2 &= -a_{21} x_2 + (a_{11} - a_{22}) x_2 x_3 + a_{12} x_2^2 x_3 + x_2^3 \\
\dot{x}_3 &= x_3 (a_{11} x_2 + a_{12} x_2 x_3 + x_2^2)
\end{align*} \]

For \( a_{21} \neq 0 \) this system has a node at \((0,0)\). This follows, for example, from Theorem 66 of [5]. The existence of a node at infinity for a system of even parity such as (5) implies the existence of unbounded trajectories (as \( t \to \infty \)). For \( a_{21} = 0 \), Equation (5) is integrable. We find that for \( a_{21} = 0 \)

\[ \begin{align*}
x_1(t) &= x_1(0) e^{a_{21} t} + e^{a_{11} t} \begin{pmatrix} a_{12} x_2(0) \\ a_{12} x_2(0) + \frac{a_{22} - a_{11}}{a_{22}} \end{pmatrix} = \begin{pmatrix} (a_{22} - a_{11}) t \\ a_{12} x_2(0) + \frac{a_{22} - a_{11}}{a_{22}} \end{pmatrix} \\
&\begin{cases} \text{if } a_{11} \neq a_{22} \\ \text{if } a_{11} = a_{22} \end{cases}
\end{align*} \]

\[ \begin{align*}
x_2(t) &= x_2(0) e^{a_{22} t}
\end{align*} \]
Thus, when \( a_{22} = 0 \), it follows that if \( a_{11} < 0 \) and \( a_{22} < 0 \) then
\[
x_2(t) \quad \text{as} \quad t \to -\infty \quad ; \quad \text{if} \quad a_{22} > 0 \quad \text{and} \quad x_2(0) \neq 0 \quad \text{then} \quad |x_2(t)| \to \infty \quad \text{as} \quad t \to -\infty \quad ; \quad \text{if} \quad a_{11} > 0 \quad \text{and} \quad x_1(0) \neq 0 \quad \text{then} \quad |x_1(t)| \to \infty \quad \text{as} \quad t \to -\infty \quad ; \quad \text{if} \quad a_{11} = 0 \quad , \quad a_{22} < 0 \quad \text{then} \quad x_2(t) \to 0 \quad \text{and} \quad x_1(t) \to \infty \quad \text{as} \quad t \to -\infty \quad ; \quad \text{if} \quad a_{11} = 0 \quad , \quad a_{22} > 0 \quad \text{then} \quad x_2(t) = x_2(0) \quad \text{and} \quad x_1(t) = -a_{22}x_2(t)/a_{11} = -x_2^2(0)/a_{11} \quad ; \quad \text{and} \quad \text{finally, if} \quad a_{11} = a_{22} = 0 \quad \text{and} \quad x_2(0) \neq 0 \quad \text{then} \quad |x_1(t)| \to \infty \quad \text{as} \quad t \to -\infty \quad . \quad \text{This completes the proof of Lemma 3.}
\]

Lastly, we consider the system (1) with \( f_2(x) \) given in B).

**Lemma 3:** The system
\[
\dot{x} = Ax + \left( -\frac{x_2^2}{x_1^2 + x_2^2} \right), \quad x(0)e^{2t}
\]
has all of its trajectories bounded (for \( t > 0 \)) if and only if
\( \|c\| > 2 \), (7) has an unbounded solution. If
\( \|c\| < 2 \) then (7) has only one critical point at infinity represented by the point \( P_1(\pm 1, 0, 0) \) on the unit sphere in \( S^3 \). In terms of local coordinates \((x_2, x_3)\) at \( P_1 \), cf. Fig. 5, Equation (7) has the form
\[
\begin{align*}
\dot{x}_2 &= -x_2^2x_3 - \left( a_{11} - a_{22} \right) x_2^2x_3 \pm ai_2 - a_{12}x_2^2x_3 - x_2^3 \\
\dot{x}_3 &= -x_3^2 \pm a_{12}x_2^2x_3 + x_2^2
\end{align*}
\]
(8)

It follows from Bendixson's theorem, cf. Lefschetz [5; p. 241] that the above system (8) has a node, a saddle or a saddle-node (two hyperbolic sectors and a par) at \((0,0)\), according to whether the index at \((0,0)\) is 0, 1, -2 or 0. Theorem 69 in Andronov and Leontovich [5; p. 379] is used.
to obtain the following results: if $a_{11} \neq 0$, then (8) has a saddle-node at $(0,0)$; if $a_{11} = 0$ and $a_{21}(a_{12} - a_{21}) > 0$, then (8) has a node at $(0,0)$; if $a_{11} = 0$ and $a_{21}(a_{12} - a_{21}) < 0$ then (8) has a saddle at $(0,0)$; if $a_{11} = 0$, $a_{12} - a_{21} = 0$ and $a_{21}(c - a_{22}) \neq 0$, then (8) has a saddle-node at $(0,0)$. The existence of the node or saddle at infinity for a system of even a parity such as (7) implies the existence of an unbounded trajectory (as $t \to -\infty$). Hence, if $a_{11} = 0$ and $a_{21}(a_{12} - a_{21}) \neq 0$, (7) has an unbounded trajectory (as $t \to -\infty$).

The behavior of (7) for $|x| > 1$; i.e., near the equator of the Poincaré sphere as determined by (8) is shown in Fig. 6 for $a_{11} < 0$ and for the case $a_{11} = 0$, $a_{12} - a_{21} = 0$ and $a_{21}(c - a_{22}) < 0$; it is shown in Fig. 7 for $a_{11} > 0$ and for the case $a_{11} = 0$, $a_{12} - a_{21} = 0$ and $a_{21}(c - a_{22}) > 0$.

Hence, all trajectories of (7) are bounded (for $t \geq 0$) if $a_{11} < 0$ or if $a_{11} = 0$, $a_{12} - a_{21} = 0$ and $a_{21}(c - a_{22}) < 0$. And (7) has unbounded trajectories (as $t \to -\infty$) if $a_{11} > 0$ or if $a_{11} = 0$, $a_{12} - a_{21} = 0$ and $a_{21}(c - a_{22}) > 0$.
If \( a_{11} = a_{22} = 0 \) then (7) has \( x_2 \) as a common factor (i.e., \( x_2 = 0 \) is a line of rest points) and the trajectories of (7) are the same as those of the related linear system (9) below for \( x_2 > 0 \) and the same as those of the related linear system (9) with the direction of motion reversed for \( x_2 < 0 \).

The related linear system

\[
\begin{align*}
\dot{x}_1 &= a_{12} + x_2 \\
\dot{x}_2 &= a_{22} - x_1 + cx_2
\end{align*}
\]

(9)

has a focus or a center at \( (a_{22} - ca_{12}, -a_{12}) \) according to whether \( c \) is equal to zero or not. All trajectories of (7) are bounded for \( t > 0 \) in this case (cf. Fig. 4 or 8).

![Fig. 8](image)

If \( a_{11} = 0 \), \( a_{22} \neq 0 \) and \( a_{12} + a_{21} = ca_{21} + a_{22} = 0 \) then (7) has \( (x_2, a_{12}) \) as a common factor (i.e., \( x_2 = -a_{12} \) is a line of rest points) and the global behavior of (7) follows from the related linear system.
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + cx_2
\end{align*}
\] (10)

which has a focus or a center at \((0,0)\) according to whether \(c\) is equal to zero or not. The behavior is equivalent to that described above. This completes the proof of Lemma 4.

The above results are summarized in the following theorem.

**Theorem 1:** The quadratic system \((1)\) (with \(f_2(x) \neq 0\)) has all of its trajectories bounded for \(t > 0\) if and only if there exists a linear transformation which reduces it to one of the systems \((3), (5),\) or \((7)\) satisfying the conditions of Lemmas 3, 5 or 4 respectively.

**Theorem 1** has an equivalent geometrical interpretation:

**Theorem 1':** The quadratic system \((1)\)

\[\dot{x} = Ax + f_2(x), \quad x(0)e^{t}\]

(with \(f_2 \neq 0\)) has all of its trajectories bounded for \(t > 0\) if and only if one of the following sets of conditions are satisfied:

i) the system \(\dot{x} = f_2(x)\) has one line of rest points, \(L\), and all other trajectories of this system are straight lines parallel to \(L\); and the matrix \(A\) has an eigenvector in the direction of \(L\), \(\det A > 0\) and \(\text{tr} A < 0\).

ii) the system \(\dot{x} = f_2(x)\) has two lines of rest points \(L\) and \(L'\) and all other trajectories of this system are half-lines parallel to \(L\);
and the matrix $\tilde{A}$ has one negative eigenvalue and one non-positive eigenvalue with the eigenvector corresponding to this latter eigenvalue in the direction of $L$.

iii) the system $\dot{x} = f_2(x)$ has one line of rest points $L$ and all other trajectories of this system are arcs of ellipses or contracting spirals tending to $L$; and either the system (1) has a line of rest points or the solutions of the system $\dot{x} = Ax$ have a decreasing radial component on $L$ or the solutions of the system $\dot{x} = Ax$ have an inward flow across the ellipses or spirals of $\dot{x} = f_2(x)$.

2. Centers for Bounded Quadratic Systems

Necessary and sufficient conditions for the existence of a center for a quadratic system in the plane are given in the survey paper of Coppel [1; p. 295]. These results are used to determine necessary and sufficient conditions for the existence of a center for systems (3), (5), and (7).

A system of the form (1) has a center at $(0,0)$ only if the origin is a center for the corresponding linear system; i.e., only if $a_{11}a_{22} - a_{12}a_{21} > 0$ and $a_{11} + a_{22} = 0$. This implies that $a_{21} \neq 0$ and therefore

$$B = \begin{bmatrix} -1 & a_{11} \\ 0 & a_{21} \end{bmatrix}$$

is a non-singular matrix with the property that

$$B^{-1}AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$ 

With $\dot{x} = By$, Equation (1) becomes

$$\dot{y} = B^{-1}ABy + B^{-1}f_2(By)$$

\[4.12\]
which is equivalent to Equation (3), p. 295 in [1]. The algebraic conditions of [1; p. 295] which are necessary and sufficient for the above system to have a center at (0,0) then yield the following results for the systems (3), (5) and (7) respectively:

**Lemma 5:** The system (3) has a center at (0,0) if and only if $a_{11} = a_{22} = 0$ and $a_{21}a_{12} < 0$.

**Lemma 6:** The system (5) has a center at (0,0) if and only if $a_{11}a_{22} - a_{12}a_{21} > 0$ and $a_{11} + a_{22} = 0$.

**Lemma 7:** The system (7) has a center at (0,0) if and only if $c = a_{11} = a_{22} = 0$ and $a_{12}a_{21} < 0$.

The above results will be used in the next section devoted to determining all possible phase-portraits for bounded quadratic systems.

### 3. Phase Portraits for Bounded Quadratic Systems

Markus [6] has shown, among other things, that two quadratic systems of the form (1) with only a finite number of rest points are topologically equivalent (i.e., there exists an orientation preserving homeomorphism, o-homeomorphism, of $E^2$ onto itself carrying trajectories of one onto trajectories of the other in a one-to-one manner) if and only if their separatrix configurations (the set of separatrices $\Sigma$ together with one trajectory from each component of $E^2 - \Sigma$) are o-homeomorphic under an o-homeomorphism of $E^2$ which is an o-homeomorphism of the corresponding sets of separatrices. When the system (1) has only a finite number of separatrices in $E^2$ as well as at infinity, a trajectory of (1) which is not a rest point is a separatrix in the sense of Markus if and only if it
contains an orbitally unstable semi-trajectory on \( S^2 \) (the unit sphere in \( E^3 \)); cf. e.g., Kalcherik, [7]. A rest point is also a separatrix of (1) in the sense of Markus.

In this section, we shall determine all possible phase-portraits for bounded quadratic systems; i.e., we shall determine all possible separatrix configurations of the systems (3), (5) and (7) subject to the conditions of Lemmas 2, 3 and 4. This amounts to finding all of the critical points in \( S^2 \) and all of those trajectories which contain an orbitally unstable semi-trajectory in \( S^2 \) (i.e., limit cycles and trajectories which form the boundary of a hyperbolic sector in \( S^2 \)); and then deducing all of the possible separatrix configurations for these systems. The number and relative position of the rest points for quadratic systems is easily deduced. The local behavior near each rest point follows from standard theorems on perturbation of linear systems in the non-degenerate case when the determinant of the linear terms does not vanish, cf. [3], and from the theorems of Andronov and Leontovich [5] in the degenerate case when the determinant of the linear terms vanishes. This will determine the number of separatrices issuing from or tending toward each rest point. Tung Chin-chu [8] showed that each limit cycle of (1) contains exactly one rest point in its interior; cf. Theorem 2, p. 296 in [1]. The number of limit cycles around each isolated rest point of (3), (5) or (7) (this number is known to be at most three, cf. [1; p. 295]), will not be taken up in this paper, since only partial results are available at this time. All possible separatrix configurations of bounded quadratic systems can still be deduced and the question of the existence and number of limit cycles around each rest point can then be taken up at a later time.

Thus, the symbol \( \rightarrow \) is used in this paper to denote either a stable node or focus or a stable or unstable focus on the interior of one, two or three limit cycles, the outermost of which is externally stable. The results compiled in [1] (in particular Theorem 6, p. 259) are being tacitly assumed in making this definition. We also interpret this symbol to include one trajectory from each component on the interior of any existing limit cycle. The symbol \( \rightarrow \) is similarly defined, the word unstable replacing the word stable in the above definition. Finally, deducing all possible separatrix configurations based on the above information is accomplished with the aid of Tung Chin-chu's results contained in the lemma on p. 296 of [1] and the Poincaré-Bendixson theorem.
Let us first consider the system (7) under the conditions of Lemma 4.

This is the most interesting case of a bounded quadratic system since the bounded cases of (3) and (4) are integrable. The critical point at infinity \( P_{\infty}(1,0,0) \) for system (7) with \(|c| < 2\) has been discussed in the proof of Lemma 4. Regarding the finite critical points of (7), we have the following lemma. First, let us define \( d = a_{11}a_{22} - a_{12}^2 \) and \( b = a_{12} - a_{21} + ca_{11} \).

**Lemma 8:** If \( a_{11} \neq 0 \), then (7) has: three (finite) rest points iff \( d \neq 0 \) and \( b^2 > 4d \); two (finite) rest points iff \( b \neq 0 \) and either \( b^2 = 4d \) or \( d = 0 \); one (finite) rest point iff either \( b^2 < 4d \) or \( b = d = 0 \).

In the case of three (finite) rest points, \( Q_i(x_1^i, x_2^i) \) \( i = 1, 2, 3 \), it follows that (with proper indexing): \( x_2^1 < x_2^2 < x_2^3 \); \( Q_2 \) is a saddle, and \( Q_1 \) and \( Q_3 \) are either nodes or foci; \( Q_3 \) lies to the right of the line \( \frac{x_2}{x_2^1} \) if \( a_{11} < 0 \).

Conversely, if \( a_{11} \neq 0 \) and \( f(7) \) has a saddle at one of its (finite) rest points, it has two other rest points, one above and one below the saddle (in the \( x_2 \)-sense) which are nodes or foci.

**Proof:** If \( a_{11} \neq 0 \) then \( Ax + (x_2^2, -x_1x_2 + cx_2) = 0 \) has the solutions \( x_1 = x_2 = 0 \) and

\[
x_1^\pm = \frac{-a_{12} \pm x_2^\pm}{a_{11}}
\]

with \( x_2^\pm \) given by

\[
x_2^\pm = \frac{-b \pm \sqrt{b^2 - 4d}}{2}
\]

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The results on the number of (finite) rest points follow immediately. It also follows from the above equations for the rest points that
\[ x_2^- < 0 < x_2^+ \text{ iff } d < 0 \text{ i.e., iff the origin is a saddle; that } \\
\begin{align*}
x_2^- < x_2^+ \text{ and } b > 0 \text{ and that } 0 < x_2^- < x_2^+ \text{ iff } d > 0 \text{ and } b < 0. 
\end{align*}
This implies the results stated for the case of three rest points (\(d \neq 0, b^2 > 4d\)): first that the \(x_2\)-coordinates of the rest points are ordered and then (by translating the origin to each of the three rest points, a transformation which leaves the form of the equation as well as the coefficient \(a_{11}\) invariant) that the middle rest point (in the \(x_2\)-sense) must be a saddle and the upper and lower rest points must be nodes or foci. This follows since the above results imply that the coefficient matrix of the linear terms at the upper and lower rest points must have a positive determinant and since these rest points cannot be centers for (7) with \(a_{11} \neq 0\) according to Lemma 7.

The fact that \(\bar{q}_1\) lies to the right of the line \(\overline{0\bar{s}_1}\) when \(a_{11} < 0\) follows from the fact that with the saddle at \(0\) the line \(\overline{q_1\bar{q}_1}\) intersects the positive \(x_1\)-axis; i.e., the equation of the line \(\overline{q_1\bar{q}_1}\)

\[ (x_2-x_2^-) = \left(\frac{x_2^+-x_2^-}{x_1^+-x_1^-}\right) (x_1-x_1^-) \]

with \(x_2 = 0\) yields (with \(a_{11} < 0\) and \(d < 0\))

\[ x_1 = \frac{x_2^+x_1^- - x_2^-x_1^+}{x_2^+-x_2^-} > 0. \]

This follows since \(x_2^+ > x_2^+\) and since the above equations for \(x_1\) imply that \(x_1^+x_1^- - x_2^+x_1^- > 0\) for \(a_{11} < 0\) and \(d < 0\).
In the bounded cases of (7) not covered in Lemma 6, we have only the one
rest point at the origin if \( a_{11} = a_1c + a_{21} = 0 \) and \( c \neq -a_2 + a_{22} < 0 \) and
a line of rest points if \( a_{11} = a_{21} = 0 \) of if \( a_{11} = a_1b + a_{21} = c a_{21} + a_{22} = 0 \). We next determine all possible separatrix configurations in
the case of three finite rest points for the bounded cases of system (7);
i.e., for \( |c| < 2 , a_{11} < 0 , d \neq 0 \) and \( b^2 > 4d \). It is no restriction
to assume that the saddle is at the origin since translating the origin
to any rest point of (7) leaves the form of the equation as well as \( a_{11} \)
invariant; i.e., we assume that \( d < 0 \). Then it is no restriction to
further assume that \( a_{21} = 0 \) since the transformation \( x_1 \rightarrow x_1 \), \( x_2 \rightarrow -x_2 \)
transforms (7) into

\[
\begin{align*}
\dot{x}_1 &= a_1 x_1 - a_2 x_2 + x_2^2 \\
\dot{x}_2 &= -a_{21} x_1 + a_{22} x_2 - x_1 x_2 - c x_2^2
\end{align*}
\]

which leaves the form of Equation (7) as well as \( a_{11} \) and \( d \) invariant.
Note, however, that this transformation is not orientation preserving.

**Lemma 9:** The separatrix configuration for the system (7) with \( |c| < 2 , a_{11} < 0 , a_{21} = 0 \), and \( d < 0 \) is \( \alpha \)-homeomorphic to one of the
configurations shown in Fig. 9.
Proof: It was shown in the proof of Lemma 4 that for \( |c| < 2 \) and
\[ a_{11} < 0 \]
the only critical point at infinity is the saddle-node at
\[ P_1(t_1, 0, 0). \]
The behavior near infinity, i.e., near the equator of the
Poincaré sphere, as determined by Equation (8) is shown in Fig. 6 for
this case. Note that the equator consists of separatrices. If \( a_{21} = 0 \)
and \( a_{11} < 0 \) the \( x_1 \)-axis is composed of trajectories with \( x_1 < 0 \) for
\( x_1 < 0 \) and \( x_2 > 0 \) for \( x_1 < 0 \). If \( d < 0 \), the \( x_1 \)-axis consists of
three separatrices (including the origin) with \((0,0)\) as their \( \omega \)-limit
set. If \( d < 0 \) there are two separatrices (besides the origin) with the
saddle at the origin as their \( \alpha \)-limit set, one in the upper half plane
\( T_1 \) and one in the lower half plane \( T_2 \). If \( d < 0 \) there is one rest
point in the upper half plane and one in the lower half plane (Lemma 8).
The Poincaré-Bendixson theorem then implies that either the rest point
in the upper half plane or an externally-stable limit cycle around that
rest point is the \( \omega \)-limit set of \( T_1 \) and that the rest point in the lower
half plane or an externally-stable limit cycle around that rest point is
the \( \alpha \)-limit set of \( T_2 \). This completes the proof of Lemma 9, the only
possible separatix configurations for this case being \( \alpha \)-homeomorphic to
one of those in Fig. 9.

Lemma 10: The separatix configuration for the system (7) with \( |c| < 2 \),
\[ a_{11} < 0, \quad a_{21} < 0, \] and \( d < 0 \) is \( \alpha \)-homeomorphic to one of the
configurations shown in Fig. 10.
Proof: It follows from Lemma 2 that under the given hypothesis, (7) has three (finite) rest points; one at the origin 0, one in the upper half plane \( Q_1 \) and one in the lower half plane \( Q_2 \) to the right of the line \( \overline{Q_1} \). According to the result of Tung Chin-chu given on p. 296 of [1], the flow on the line \( \overline{Q_2} \) is in the same sense on the segments \( \overline{Q_2} \) and \( \overline{Q_2} \) and in the opposite sense on the segment \( \overline{Q_2} \).

Similar results follow for the flow on \( \overline{Q_2} \), cf. Fig. 11. The flow on the \( x_1 \)-axis for \( a_{21} < 0 \) is c.w. (except at \( x_1 = 0 \)).

![Fig. 11](image)

The \( x_1 \)-axis and the lines \( \overline{Q_1} \), \( \overline{Q_2} \) divide \( S^2 \) into six sectors \( R_i \), \( i = 1, \ldots, 6 \) numbered with increasing argument. The flow on the \( x_1 \)-axis and on the lines \( \overline{Q_1} \) and \( \overline{Q_2} \) near the origin establishes the fact that (with \( d < 0 \)) the two separatrices which have the saddle at the origin as their \( \alpha \)-limit set. \( T_1 \) and \( T_2 \), approach \((0,0)\) as \( t \to -\infty \) in the sectors \( R_1 \) and \( R_4 \) respectively; and that the two separatrices which have \((0,0)\) as their \( \alpha \)-limit set, \( T_3 \) and \( T_6 \), approach \((0,0)\) as \( t \to +\infty \) in the sectors \( R_2 \) and \( R_5 \) respectively. It follows from the
Equations (8) describing the behavior near $P_1$ that for $\alpha_{11} < 0$ the separatrix $T_4$ with $P_1$ as its $\alpha$-limit set approaches $P_1$ as $t \to -\infty$ in sector $R_6$; cf. Fig. 11.

Now since the boundary of $R_6$ as a subset of $S^2$ consists of points of egress (the lines $\partial R_1$ and $\partial R_6$), a trajectory (on the equator), a saddle at $0$ and a saddle-node $P_1$, it follows from the Poincaré-Bendixson theorem that the saddle-node $P_1$ must be the $\alpha$-limit set of $T_2$ as is shown in Fig. 11. Similarly, considering the boundary of $R_4 \cup R_6$, it follows that $Q_3$ or an externally stable limit cycle around $Q_3$ must be the $\alpha$-limit set of the separatrices $T_4$ and $T_5$ is shown in Fig. 11. Finally, considering the boundary of $R_1 \cup R_2 \cup R_6 \cup R_6$, it follows from the Poincaré-Bendixson theorem that the separatrix $T_2$ must have one of the following sets for its $\alpha$-limit set:

1) the rest point $Q_1$ or an externally-stable limit-cycle about $Q_1$, in which case $T_1$ has $P_1(-1,0,0)$ as its $\alpha$-limit set, as in Fig. 10(a),

2) the saddle at $(0,0)$; i.e., $T_3$ coincides with $T_3$ and is a separatrix-cycle which contains the rest point $Q_2$ (cf. Theorem 2, p. 296 in [1]) as well as any limit cycle around $Q_1$, as in Fig. 10(c), (d) or

3) the rest point $Q_3$ or an externally-stable limit cycle around $Q_3$, in which case $T_1$ has $Q_3$ or an externally-unstable limit cycle around $Q_3$ as its $\alpha$-limit set, as in Fig. 10(b).

Note that in the case when $T_1$ is a separatrix cycle, the rest point $Q_3$ (in the interior of $T_1 \cup \{0\}$) is either a stable or an unstable focus (cf. Theorem 6, p. 299 in [1]), since according to Lemma 7 it is not a center. Also, if $T_1$ is a separatrix cycle and there is a limit cycle around $Q_2$ (in the interior of $T_1 \cup \{0\}$), the outermost limit cycle around $Q_2$ can be either externally-stable or externally-unstable. All of the above separatrix configurations are summarized in Fig. 10 using the $\rightarrow 0$ symbol. This completes the proof of Lemma 10.
Comment 1: In the cases of Lemma 10 when there is no limit cycle about the upper rest point $Q_1$, it follows that the cases when $Q_1$ is stable correspond to either Fig. 10(a) or (c) and the cases when $Q_1$ is unstable to either Fig. 10(b) or (d).

Comment 2: Numerical results obtained on an analog computer indicate that the cases of Figures 10(c) and (d) in which a separatrix cycle appears correspond to the cases when $Q_1$ is a weak focus.

Comment 3: We have obtained numerical examples of all of the possible configurations of Figs. 9 and 10 with fewer than two cycles, but have not been able to distinguish the cases corresponding to Fig. 10(c) from those corresponding to 10(d).

Comment 4: We note that Lemma 10 also establishes all possible separatrix configurations for (7) in case $|c| < 2$ and $a_{11} > 0$. This follows since under the transformation $x_1 = -x_1$, $x_2 = x_2$, $t = -t$ (7) becomes

$$\dot{x}_1 = -a_{11}x_1 + a_{12}x_2 + x_2^2$$
$$\dot{x}_2 = a_{21}x_1 - a_{22}x_2 - x_1x_2 - cx_2^2$$

And the form of (7) as well as $a_{21}$ and $d$ are left invariant under this transformation.

We next determine all possible separatrix configurations in the case of two (finite) rest points for the bounded cases of system (7); i.e., for $|c| < 2$, $a_{11} < 0$, $b \neq 0$ and either $d = 0$ or $b^2 = 4d$. We may assume that $d = 0$, for if $d \neq 0$ and $b^2 = 4d$ then translating the origin to the second rest point yields a system of the same form as (7) with $a_{11} < 0$ left invariant and with the determinant of the transformed matrix $A$ equal to zero. We also note that $d = 0$ and $|c| < 2$ imply that $b \neq 0$.

The same type of argument used in Lemma 10 can be employed in this case.
However, it is more straightforward to observe that due to the continuity of solutions with respect to the parameter $d$, letting $d \to 0$ through negative values will result in a continuous deformation of the separatrix configurations in Figs. 9 and 10 with one of the rest points $Q_2$ or $Q_3$ approaching 0 as $d \to 0$. The nature of the resulting degenerate rest point at 0 (with $d = 0$) can be investigated using Theorem 55 of [5] if $a_{11} + a_{22} \neq 0$ and Theorem 67 of [5] if $a_{11} + a_{22} = 0$. The results are just as one would expect: 0 is a saddle-node if $a_{11} + a_{22} \neq 0$ and $d = 0$ (this also follows from Bendixon's theorem since the index on any sufficiently large circle is equal to the sum of the indices at the finite rest points and is equal to +1 in this case); and 0 is a degenerate rest point consisting of two hyperbolic sectors if $a_{11} + a_{22} = d = 0$.

The separatrix configurations resulting when either $Q_2$ or $Q_3$ approaches 0 in the cases of Fig. 9 are shown in Figs. 12(a) and (b) respectively. The separatrix configurations resulting when $Q_2$ approaches 0 in the cases of Fig. 10(a) are shown in Fig. 13(a) if $\lim_{d \to 0} (a_{11} + a_{22}) \neq 0$ and in Fig. 14 if $\lim_{d \to 0} (a_{11} + a_{22}) = 0$; and in the cases of 10(b) they are shown in Fig. 13(b) if $\lim_{d \to 0} (a_{11} + a_{22}) \neq 0$ and in Fig. 14 if $\lim_{d \to 0} (a_{11} + a_{22}) = 0$; and in the cases of 10(c) or (d) they are shown in Fig. 14. The separatrix configurations resulting when $Q_3 \to 0$ in the cases of 10(a), (b), (c), and (d) are shown in Fig. 13(c), (d), (e) and (f) respectively. These results are summarized in the following lemmas:

**Lemma 11:** The separatrix configuration for system (7) with $|c| < 2$,

- $a_{11} < 0$, $a_{22} = a_{22} = 0$ and 1) $a_{12} + c a_{11} < 0$ is $C^1$-homeomorphic to one of the configurations shown in Fig. 12(a); ii) $a_{12} + c a_{11} > 0$ is $C^1$-homeomorphic to one of the configurations shown in Fig. 12(b).
Lemma 12: The separatrix configuration for system (7) with $|c| < 2$, $a_{11} < 0$, $a_{21} < 0$, $a_{11}a_{22} = a_{12}a_{21}$, $a_{11} + a_{22} \neq 0$ and 1) $a_{12} - a_{21} + ca_{11} < 0$ is $C^\infty$-homeomorphic to one of the configurations in Figs. 13(c) - (f); 2) $a_{12} - a_{21} + ca_{11} > 0$ is $C^\infty$-homeomorphic to one of the configurations in Figs. 13(a) and (b).

Fig. 13

Lemma 13: The separatrix configuration for system (7) with $|c| < 2$, $a_{11} < 0$, $a_{21} < 0$, $a_{11}a_{22} = a_{12}a_{21}$ and $a_{11} + a_{22} = 0$ is $C^\infty$-homeomorphic to one of the configurations shown in Fig. 14.

Fig. 14

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Note that the hypotheses of Lemma 13 imply that \( b > 0 \) (and therefore that \( q_1 = 0 \) and \( q_2 \neq 0 \) for this case).

**Comment 5:** In the case of no limit cycle around the upper rest point, the cases corresponding to Figs. 15(c) - (f) may be distinguished as in Comment 1 according to the stability of \( q_1 \).

We next determine the possible separatrix configurations in the case of one (finite) rest point at the origin for the bounded cases of system (7); i.e., for \( |c| < 2 \) and either i) \( a_{11} = 0, a_{21} \neq 0, a_{21} + a_{22} = 0 \) and \( ca_{21} + a_{22} < 0 \) in which case the origin is a stable node (for \( a_{22} < 0 \), \( -a_{22} > 2|a_{21}| \)), a stable focus (for \( a_{22} < 0 \), \( -a_{22} < 2|a_{21}| \)) or (by the Poincaré-Bendixson theorem) an unstable focus in the interior of a limit cycle (for \( a_{22} > 0 \), a limit cycle being generated at the origin as \( a_{22} \) becomes positive); or ii) \( a_{11} < 0 \) and \( b^2 < 4d \) in which case the origin is a stable node or focus (for \( a_{11} + a_{22} = 0 \)) or (by the Poincaré-Bendixson theorem) an unstable focus in the interior of a limit cycle (for \( a_{11} + a_{22} > 0 \), a limit cycle being generated at the origin as \( a_{11} + a_{22} \) becomes positive).

In these cases, it was pointed out in the proof of Lemma 4 that (7) has only one critical point at infinity, a saddle-node as shown in Fig. 6.

The above remarks establish the following lemma.

**Lemma 14:** The separatrix configuration for system (7) with \( |c| < 2 \), and either i) \( a_{11} = a_{12} + a_{21} = 0, a_{21} \neq 0 \) and \( ca_{21} + a_{22} < 0 \) or ii) \( a_{11} < 0 \) and \( b^2 < 4d \) is \( \alpha \)-homeomorphic to one of the configurations in Fig. 15.

![Fig. 15](image-url)
The global behavior of the bounded cases of system (7) when there exists a line of rest points is easily deduced from the related linear system (9) in case $a_{11} - a_{21} = 0$ or (10) in case $a_{11} = a_{21} + a_{12} = a_{22} = 0$, $a_{21} \neq 0$ and the global behavior can be depicted pictorially as in Fig. 4 or 8 (this is not to say that these figures determine the topological behavior of (7) in this case as do the separatrix configurations of Figs. 9 - 15).

We next consider the bounded cases of system (3). The bounded cases of (3) are integrable and the solution of (3) in these cases is given in Equation (4). It is useful in determining the global behavior to note that there are two critical points at infinity, $P_1(\pm 1, 0, 0)$ and $P_2(0, \pm 1, 0)$, and that it follows from Theorem 65 of [5] that $P_1$ is a saddle-node if $a_{11} \neq 0$. If $a_{22} = 0$ the $x_2$-axis is a line of rest points and the trajectories of (3) coincide with those of the related linear system

$$
\begin{align*}
\dot{x}_1 &= a_{11} \\
\dot{x}_2 &= a_{21}x_2
\end{align*}
$$

(11)

for $x_2 > 0$ and with the trajectories of (11) with the direction of motion reversed for $x_2 < 0$. The global behavior in this case is depicted pictorially in Fig. 16.

Fig. 16

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If \( a_{12} = 0, a_{11} < 0 \) and \( a_{22} < 0 \) the local behavior near \( P_2 \) is not determined by the results of [5]. However, it follows from the solution (4) that \( P_2 \) is a saddle-node (this follows since the \( x_2 \)-axis and the equator of \( S^2 \) consist of trajectories, every trajectory in \( x_1 > 0 \) has the origin as its \( \omega \)-limit set and \( P_1 \) as its \( \alpha \)-limit set and every trajectory in \( x_1 < 0 \) has the origin as its \( \omega \)-limit set and \( P_2 \) as its \( \alpha \)-limit set). Also in this case, the origin is a stable node. Thus, we have the following lemma.

**Lemma 15:** The separatrix configuration for system (3) with \( a_{12} = 0, a_{11} < 0 \) and \( a_{22} < 0 \) is \( \sigma \)-homeomorphic to the configuration shown in Fig. 17.

![Fig. 17](image)

We finally consider the bounded cases of (5) which are also integrable. The solution is given in Equations (6). If \( a_{21} = a_{22} = 0 \) we have a parabola of rest points, \( x_2^2 + a_{12}x_2 + a_{11}x_1 = 0 \) and the global behavior in this case is depicted pictorially in Fig. 18 for \( a_{11} < 0 \).

![Fig. 18](image)
If $a_{21} = 0$, $a_{22} < 0$ and $a_{11} < 0$ then the $x_1$-axis consists of trajectories, the origin is a stable node and it follows from (6) that the critical point at infinity $p_1(i1, 0, 0)$ is a saddle-node. We have the following lemma.

**Lemma 16:** The separatrix configuration for system (5) with $a_{21} = 0$, $a_{22} < 0$ and $a_{11} < 0$ is $o$-homeomorphic to the configuration in Fig. 15, with $-\infty$ interpreted as a stable node.

We summarize these results in the following theorem wherein a phase portrait means an equivalence class of the trajectories of a system, two sets of trajectories of a system in $\mathbb{R}^2$ being equivalent iff there exists a homeomorphism of $\mathbb{R}^2$ carrying trajectories of one onto trajectories of the other in a 1:1 manner.

**Theorem 2:** The phase portrait of a bounded quadratic system is either determined by one of the separatrix configurations shown in Figs. 9, 10, 12-15, 17, or it is determined by quadratures as in the cases depicted in Figs. 4, 8, 16, 18.

The definition of phase-portrait based on equivalence under homeomorphism could just as well be based on equivalence under $o$-homeomorphism provided the phase-portraits obtained by rotating those of Figs. 8, 10, 13 and 14 about the $x_1$-axis are included in the glossary of possible types in Theorem 2.

We note that in the class of bounded quadratic systems, the structurally-stable systems have separatrix configurations homeomorphic to one of those in Fig. 10(a), 10(b), 15 or 17 under a homeomorphism of $\mathbb{R}^2$.

This section classifies bounded quadratic systems in the plane (Theorem 1) and determines all possible phase-portraits for such systems (Theorem 2). It is also a step in the direction of characterizing the phase-portraits of all bounded quadratic systems in the plane by means of algebraic in-
equalities on the coefficients (Lemmas 9 - 16). In order to complete this type of algebraic classification, it is necessary to determine the number and stability properties of the limit cycles around each isolated rest point. This information together with a means of determining the stability properties of each fixed should be sufficient to complete the algebraic classification problem.

Section 4 References


8. Tung Chin-chu, "Positions of limit cycles of the system

\[ \frac{dx}{dt} = \sum_{0 \leq i + k \leq 2} a_{ik} x^i y^k, \quad \frac{dy}{dt} = \sum_{0 \leq i + k \leq 2} b_{ik} x^i y^k, \]

Section 5
CRITICAL POINTS OF QUADRATIC SYSTEMS IN $\mathbb{R}^2$

The classification of two-dimensional homogeneous quadratic systems

$$\dot{x} = f_2(x), \quad x \in \mathbb{R}^2$$

(1)

given by Markus [1] establishes a natural starting point for classifying two-dimensional quadratic systems in general. A two-dimensional quadratic system can be written in the form

$$\dot{x} = k + Ax + f_2(x), \quad x \in \mathbb{R}^2$$

(2)

where $k = (k_1, k_2)^T$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $i, j = 1, 2$ and $f_2(x)$ is a homogeneous quadratic polynomial in $(x_1, x_2)$.

It follows from Markus' Theorem 6 [1, p. 194], that if (1) has a line of rest points then $f_2(x)$ is linearly equivalent to one of the following forms:

I. A) $\begin{pmatrix} 0 \\ c \end{pmatrix}$, $-\infty < c < \infty$

B) $\begin{pmatrix} 0 \\ x_2 \end{pmatrix}$

C) $\begin{pmatrix} x_1 x_2 + x_2^2 \\ x_2^2 \end{pmatrix}$

D) $\begin{pmatrix} -x_2^2 \\ 1 \cdot x_2 + cx_2 \end{pmatrix}$, $-\infty < c < \infty$

E) $\begin{pmatrix} x_2^2 \\ 1 \cdot x_2 + cx_2 \end{pmatrix}$, $-\infty < c < \infty$
It follows from Theorem 7 in [1; p. 196] that if (1) has no line of rest points and has two or three ray solutions then \( f_2(x) \) is linearly equivalent to one of the following forms:

**II. A)**

\[
\begin{pmatrix}
\alpha x_1^2 \\
\alpha x_2^2
\end{pmatrix}, \quad 0 < \alpha \leq 1/2 \quad \text{or} \quad \alpha \leq 1
\]

**B)**

\[
\begin{pmatrix}
\alpha x_1^2 x_2^2 \\
\alpha x_2 x_1 x_2^2
\end{pmatrix}, \quad 0 < \alpha \leq 1/2 \quad \text{or} \quad \alpha \leq 1
\]

**C)**

\[
\begin{pmatrix}
\alpha x_1^2 x_2^2 \\
\alpha x_2 x_1 x_2^2
\end{pmatrix}, \quad \text{either 1) } \alpha = 1, \beta \neq 0, \beta \neq 1 \quad \text{or} \quad 2) \alpha > 1, 0 < \beta < 1, \alpha \beta \neq 1 \quad \text{or} \quad 3) \alpha < -1/2, 0 < |\beta| < 1
\]

It follows from Theorem 8 in [1; p. 200] that if (1) has no line of rest points and has one ray solution then \( f_2(x) \) is linearly equivalent to one of the following forms:

**III. A)**

\[
\begin{pmatrix}
\alpha x_1^2 x_2^2 \\
\alpha x_1 x_2^2
\end{pmatrix}, \quad 0 < \alpha < 1, \alpha \beta < 2\sqrt{\frac{1-\alpha}{2}}
\]

**B)**

\[
\begin{pmatrix}
\alpha x_1^2 x_2^2 \\
\alpha x_1 x_2 x_2^2
\end{pmatrix}, \quad 0 < \alpha < 1, 0 < \beta < 2\sqrt{\frac{1-\alpha}{2}}
\]

**C)**

\[
\begin{pmatrix}
\alpha x_1^2 x_2^2 \\
\alpha x_1 x_2^2
\end{pmatrix}
\]
D) \[ \left( \frac{x_1^2 - x_2^2}{\alpha^2_1 x_1^2 - \beta^2_2 x_2^2} \right), \quad \alpha > 1, \quad \beta \neq \alpha, \quad 0 < \beta < 2^{\frac{1}{\alpha - 1}} \]

E) \[ \left( \frac{x_1^2 + 2x_1 x_2 + \alpha x_2^2}{2x_1 x_2} \right), \quad \alpha < -1 \]

Furthermore, it follows from [1] that any two-dimensional quadratic form \( f_2(x) \) is linearly equivalent to one of the above forms; thus, it is no restriction in determining the possible phase-portraits for (2) to assume that \( f_2(x) \) in (2) has one of the above forms. The distribution of the critical points of (2) on the equator of the Poincaré sphere is determined by the distribution of critical points of (1) on the Poincaré sphere. The nature of the critical points at infinity for (2) can be determined from the nature of the critical points at infinity for (1) in all cases except when the critical point lies at the "end" of a line of rest points of (1). (This can only happen when \( f_2(x) \) is given by one of the forms in I.) This can be shown by writing (1) and (2) in terms of local coordinates at any critical point on the equator of \( S^2 \), not at the end of a line of rest points of (1) and determining the nature of the rest point using the results of [2] or [3]. In this regard, the "phase-portraits" of (1) with \( f_2(x) \) given by one of the forms in I are shown in Fig. 1; with \( f_2(x) \) given by one of the forms in II, they are shown in Fig. 2 and with \( f_2(x) \) given by one of the forms in III, they are shown in Fig. 3. Note that case II. C) with \( \alpha < -1/2 \) and \( -1 < \beta < 0 \) implies three nodes at infinity rather than two nodes and a saddle as is stated by Markus in Theorem 9-I-3b [1; p. 206]. Also, the list of possible "phase-portraits" for homogeneous quadratic systems in [1; p. 209] is missing this case. That this is one of the possible portraits for a homogeneous quadratic system was noted earlier in the work of Lyapunov [4]. Thus, the nature of the critical points at infinity of (2) which do not lie at the end of a line of rest points of (1) is determined. The nature of the critical points at infinity in the remaining cases can be determined by writing (2) in local coordinates at the critical point at infinity and applying either classical theorems or
the results of [3] on the local behavior near critical points (or in those
degenerate cases not covered in [3] by finding the zeros of $s$ and de-
determining the sign of $f$ on these rays). Thus, the distribution and nature
of the critical points at infinity for (2) has been determined. The
results for those cases when $k^2 + b_1 x_1^2 + b_2 x_2^2 = h(x)$ and $k^2 + b_1 x_1^2 + b_2 x_2^2 = h(x)$
$f_2(x)$ have no common factor are listed in Tables 1-3. The distribution
and nature of the critical points at infinity has been studied in those
cases when (2) has a center by Latipov [5] and Lukasivic [6]. Also
Latipov [7] determines the distribution and nature of the critical points
at infinity in case $q(x) = 0$ has three simple roots (corresponding to
the non-degenerate cases of II in Table 2).

Once the distribution and nature of the critical points at infinity for
(2) with $f_2(x)$ given by II or III is known, the nature of the finite
critical points can be deduced in those cases when the number of these
points has the (maximum possible) value four. This follows from the fact
that the sum of the indices at the finite and infinite critical points is
equal to +1 and from the theorem of Jules and Casanova listed as Theorem
7 in [8]; this theorem says that in the case of four rest points we have
one of the following combinations: 2-0, 2-0; 3-0, 1-0; or 1-2, 3-0
(cf. the list of abbreviations below). These results concerning the finite
critical points are also listed in Tables 2 and 3. The nature of the finite
critical points when $f_2(x)$ is given by I and the maximum number (which is
at most three) of finite critical points occurs can be deduced by algebraic
means as in Lemma 8 of Section 6. These results are listed in Table 1.

The following list of abbreviations is used in the tables:

$D_n(f_{\lambda i j k})$ = a degenerate rest point with index $n$ consisting of $i$
fans, $j$ elliptic sectors and $k$ hyperbolic sectors

$N$ = a node = $D_{-1}(\ell)$

$S$ = a saddle = $D_{-1}(h)$

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\[ F = \text{a focus} = D_{11}(c) \]
\[ C = \text{a center} \]
\[ O = N, \ F \text{ or } C \]
\[ SN = \text{a saddle node} = D_0(\Phi_2) \]
\[ E = D_{11}(\text{eh}) \]

Section 5 References


4. Lyapunov, L. V., "The integral curves of the equation \( y' = (ax^2+bx+y+c^2)/(dx^2+ex+y^2) \)," Uspekhi Mat. Nauk 6, No. 2 (40), 171-183, (1951).


Fig. 1: Homogeneous quadratic systems with a line of rest points.
II.

(a) \(0 < \alpha \leq 1/2\)
(b) \(\alpha = 1\)
(c) \(\alpha > 1\)

(c) \(\alpha < 1/2\)
(c) \(\alpha = 1/2\), \(0 < \beta < 1\)
(c) \(\alpha > 1/2\), \(\beta > 1\)

(c) \(\alpha > 1\), \(0 < \beta < 1\)
(c) \(\alpha = 1/2\), \(-1 < \beta < 0\)

Fig. 2: Homogeneous quadratic systems with two or three ray solutions and no line of rest points.

III.

(A, B)

(c, d, e)

Fig. 3: Homogeneous quadratic systems with one ray solution and no line of rest points.

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Table 1: Critical points of (2) with $r_2(x)$ given in I.

<table>
<thead>
<tr>
<th>$r_2(x)$</th>
<th>c=0</th>
<th>$a_{12}$ $a_{12}^f_0$</th>
<th>k=0</th>
<th>3: N,1-E</th>
<th>2: 1-S,1-0</th>
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<td></td>
<td></td>
<td>$a_{11}^0, a_{12}^a_2&lt;0$</td>
<td>k=0</td>
<td>1-S,1-E</td>
<td>1: 1-0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$a_{11}^0, a_{12}^a_2&gt;0$</td>
<td>k=0</td>
<td>1-N,1-E</td>
<td>1: 1-S</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$a_{12}^0, a_{11}^a_2&lt;0$</td>
<td>k=0</td>
<td>2: SN</td>
<td>1: 1-0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$a_{11}=a_{12}=0, a_{12}^f_0$</td>
<td>k=0</td>
<td>1-SN,1-E</td>
<td>0: 1-0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$a_{22}=a_{12}=0, a_{11}^f_0$</td>
<td>k=0</td>
<td>1-SN,1-E</td>
<td>0: 1-0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$a_{11}=a_{12}=0$</td>
<td>k=0</td>
<td>1-SN,1-D 1</td>
<td>0: 1-0</td>
</tr>
<tr>
<td>c=1</td>
<td></td>
<td>$a_{12}^f_0$</td>
<td>k=0</td>
<td>1-N,1-S</td>
<td>3: 1-S,2-0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$a_{12}^f_1=0, a_{22}^f_0$</td>
<td>k=0</td>
<td>1-N,1-D 0 (SFd)</td>
<td>2: 1-S,1-0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$a_{12}^f_1=0, a_{22}^f_0$</td>
<td>k=0</td>
<td>1-N,1-D 0 (SFH)</td>
<td>2: 2-0</td>
</tr>
<tr>
<td></td>
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<td>$a_{12}^f_0=0, a_{22}^f_0$</td>
<td>k=0</td>
<td>1-N,1-D 0</td>
<td>0: 2-0</td>
</tr>
<tr>
<td>0&lt;=c</td>
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<td>$a_{12}^f_0=0, a_{22}^f_0$</td>
<td>k=0</td>
<td>1-S,1-E</td>
<td>3: 1-S,2-0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$a_{12}^f_0=0, a_{22}^f_0$</td>
<td>k=0</td>
<td>1-S,1-D 0 (FSD)</td>
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<tr>
<td></td>
<td></td>
<td>$a_{12}^f_0=0, a_{22}^f_0$</td>
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<td>1-S,1-D 0 (HFD)</td>
<td>2: 2-0</td>
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<tr>
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<td>k=0</td>
<td>1-N,1-D 0</td>
<td>0: 2-0</td>
</tr>
<tr>
<td>c=1</td>
<td></td>
<td>$a_{12}^f_0=0, a_{22}^f_0$</td>
<td>k=0</td>
<td>= i.e a line of r.p.</td>
<td>3: 1-S,2-0</td>
</tr>
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</table>

(Continued on next page)
<table>
<thead>
<tr>
<th>B</th>
<th>$a_{21}^e$0</th>
<th>$k=0$</th>
<th>1-N</th>
<th>2</th>
<th>1-S, 1-0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{21}=a_{11}a_{22}&gt;0$</td>
<td>$k=0$</td>
<td>1-S H</td>
<td>1</td>
<td>1-0</td>
<td></td>
</tr>
<tr>
<td>$a_{21}=a_{11}a_{22}&lt;0$</td>
<td>$k=0$</td>
<td>1-0</td>
<td>1</td>
<td>1-S</td>
<td></td>
</tr>
<tr>
<td>$a_{21}=a_{11}=0$</td>
<td>$k=0$</td>
<td>1-D$_{1}$</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$a_{21}=a_{11}=a_{22}=0$</td>
<td>$k=0$</td>
<td>1-P$_{41}$</td>
<td>0</td>
<td>0</td>
<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>C</th>
<th>$a_{21}^e$0</th>
<th>$k=0$</th>
<th>1-S N</th>
<th>3</th>
<th>1-S, 2-0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{21}=a_{11}(a_{11}-a_{22})&gt;0, a_{22}&lt;0$</td>
<td>$k=0$</td>
<td>1-D$_{1}$ (FE3)</td>
<td>2</td>
<td>2-0</td>
<td></td>
</tr>
<tr>
<td>$a_{21}=a_{11}(a_{11}-a_{22})&gt;0, a_{22}&lt;0$</td>
<td>$k=0$</td>
<td>1-D$_{2}$ (RE3)</td>
<td>2</td>
<td>2-S</td>
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</tr>
<tr>
<td>$a_{21}=a_{11}(a_{11}-a_{22})&lt;0, a_{22}&lt;0$</td>
<td>$k=0$</td>
<td>1-D$_{1}$ (RE3)</td>
<td>2</td>
<td>1-S, 1-0</td>
<td></td>
</tr>
<tr>
<td>$a_{21}=a_{11}=0$</td>
<td>$k=0$</td>
<td>1-D$_{41}$</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$a_{21}=a_{11}=a_{22}=0$</td>
<td>$k=0$</td>
<td>1-D$_{41}$</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

| D     | $|c|<2$ | $a_{11}^e$0 | $k=0$ | 1-S N | 3 | 1-S, 2-0 |
|-------|-------|-------------|-------|-------|---|-----------|
| $a_{11}=a_{11}(a_{11}+a_{21})>0$ | $k=0$ | 1-N | 2 | 1-S, 1-0 |
| $a_{11}=a_{11}(a_{11}+a_{21})<0$ | $k=0$ | 1-S | 2 | 2-0 |
| $a_{11}=a_{11}(a_{11}+a_{21})<0, a_{11}=a_{21}$ | $k=0$ | 1-S N | 1 | 1-0 |
| $a_{11}=a_{11}=0$ | $k=0$ | 1-N | 0 | 0 |
| $a_{11}=a_{11}+a_{21}=a_{22}=a_{21}=0$ | $k=0$ | 1-N | 0 | 0 |

| $|c|>2$ | $a_{11}^e$0 | $k=0$ | 2-S N | 3 | 1-S, 2-0 |
|-------|-------------|-------|-------|---|-----------|
| $a_{11}=a_{11}(a_{11}+a_{21})>0$ | $k=0$ | 1-N, 1-S N | 2 | 2-0 |
| $a_{11}=a_{11}(a_{11}+a_{21})<0$ | $k=0$ | 1-S, 1-S N | 2 | 2-0 |
| $a_{11}=a_{11}(a_{11}+a_{21})<0, a_{11}=a_{21}$ | $k=0$ | 2-S N | 1 | 1-0 |
| $a_{11}=a_{11}=0$ | $k=0$ | 1-N, 1-S N | 0 | 0 |
| $a_{11}=a_{11}+a_{21}=a_{22}+a_{21}=0$ | $k=0$ | 1-N, 1-S N | 0 | 0 |

(Continued on next page)
<table>
<thead>
<tr>
<th>D</th>
<th>(a_{11'} \neq 0)</th>
<th>(a_{11'} = 0, a_{21'}(a_{22'} + a_{21'}) \neq 0)</th>
<th>(a_{11'} = 0, n_{21'}(n_{22'} + n_{21'}) \neq 0)</th>
<th>(a_{11'} = 0, a_{21'}(a_{22'} + a_{21'}) = 0)</th>
<th>(a_{11'} = 0, n_{21'}(n_{22'} + n_{21'}) = 0)</th>
<th>(k = 0)</th>
<th>(1, 5, 1, 1, 1, 1, 1)</th>
<th>(k = 0)</th>
<th>(1, 5, 2, 1, 1, 1, 1)</th>
<th>(1, 5, 3, 2, 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>(a_{11} \neq 0)</td>
<td>(a_{11} = 0, a_{21}(a_{22} - a_{21}) \neq 0)</td>
<td>(a_{11} = 0, a_{21}, a_{22} \neq 0)</td>
<td>(a_{11} = 0, a_{21}, a_{22} \neq 0, a_{21} \neq 0, a_{21} &gt; a_{12})</td>
<td>(a_{11} = 0, a_{21}, a_{22} \neq 0, a_{21} \neq 0, a_{21} &gt; a_{12})</td>
<td>(k = 0)</td>
<td>(2, 5, 1, 1, 1, 1, 1)</td>
<td>(k = 0)</td>
<td>(2, 5, 2, 1, 1, 1, 1)</td>
<td>(2, 5, 3, 2, 0)</td>
</tr>
<tr>
<td>F</td>
<td>(a_{11} \neq 0)</td>
<td>(a_{11} = 0, a_{21} \neq 0)</td>
<td>(a_{11} = 0, a_{21}, a_{12} = 0)</td>
<td>(a_{11} = 0, a_{21}, a_{12} = 0, a_{21} \neq 0, a_{21} &gt; a_{12})</td>
<td>(a_{11} = 0, a_{21}, a_{12} = 0, a_{21} \neq 0, a_{21} &gt; a_{12})</td>
<td>(k = 0)</td>
<td>(1, 5, 1, 1, 1, 1, 1)</td>
<td>(k = 0)</td>
<td>(1, 5, 1, 1, 1, 1, 1)</td>
<td>(1, 5, 3, 2, 0)</td>
</tr>
<tr>
<td>G</td>
<td>(a_{11} \neq 0)</td>
<td>(a_{11} = 0, a_{21} \neq 0)</td>
<td>(a_{11} = 0, a_{21}, a_{22} \neq 0)</td>
<td>(a_{11} = 0, a_{21}, a_{22} \neq 0, a_{21} \neq 0, a_{21} &gt; a_{12})</td>
<td>(a_{11} = 0, a_{21}, a_{22} \neq 0, a_{21} \neq 0, a_{21} &gt; a_{12})</td>
<td>(k = 0)</td>
<td>(1, 5, 2, 1, 1, 1, 1)</td>
<td>(k = 0)</td>
<td>(2, 5, 1, 1, 1, 1, 1)</td>
<td>(1, 5, 3, 2, 0)</td>
</tr>
<tr>
<td>H</td>
<td>(a_{11} = 0, \sqrt{a_{12}^2 - 4a_{11}a_{22}} = 0)</td>
<td>(a_{11} = 0, \sqrt{a_{12}^2 - 4a_{11}a_{22}} = 0)</td>
<td>(a_{11} = 0, \sqrt{a_{12}^2 - 4a_{11}a_{22}} &lt; 0)</td>
<td>(a_{11} = 0, \sqrt{a_{12}^2 - 4a_{11}a_{22}} &lt; 0)</td>
<td>(a_{11} = 0, \sqrt{a_{12}^2 - 4a_{11}a_{22}} &lt; 0)</td>
<td>(k = 0)</td>
<td>(a \text{ is a line of p.p.})</td>
<td>(k = 0)</td>
<td>(1, 5, 1, 1, 1, 1, 1)</td>
<td>(1, 5, 3, 2, 0)</td>
</tr>
<tr>
<td></td>
<td>(a_{11} = 0, \sqrt{a_{12}^2 - 4a_{11}a_{22}} &lt; 0)</td>
<td>(a_{11} = 0, \sqrt{a_{12}^2 - 4a_{11}a_{22}} &lt; 0)</td>
<td>(a_{11} = 0, \sqrt{a_{12}^2 - 4a_{11}a_{22}} &lt; 0)</td>
<td>(a_{11} = 0, \sqrt{a_{12}^2 - 4a_{11}a_{22}} &lt; 0)</td>
<td>(a_{11} = 0, \sqrt{a_{12}^2 - 4a_{11}a_{22}} &lt; 0)</td>
<td>(k = 0)</td>
<td>(1, 5, 1, 1, 1, 1, 1)</td>
<td>(k = 0)</td>
<td>(1, 5, 1, 1, 1, 1, 1)</td>
<td>(1, 5, 3, 2, 0)</td>
</tr>
</tbody>
</table>

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Table 2: Critical points of (2) with \( f_2(x) \) given in II

<table>
<thead>
<tr>
<th>( f_2(x) )</th>
<th>Algebraic Constraints</th>
<th>Critical Pts. At Infinity</th>
<th>Finite Rest Pts. in the 4 r.p. Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>II. A</td>
<td></td>
<td>2-N, 1-S</td>
<td>2-S, 2-0</td>
</tr>
<tr>
<td>B</td>
<td>( \alpha \leq 0 ) or ( \alpha &gt; 1 )</td>
<td>2-N, 1-S</td>
<td>2-S, 2-0</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 1 )</td>
<td>1-N, 1-S</td>
<td>2-S, 2-0</td>
</tr>
<tr>
<td>C</td>
<td>( \alpha = 1, 0 &lt; \beta &lt; 1 )</td>
<td>1-SN, 1-S</td>
<td>1-S, 3-0</td>
</tr>
<tr>
<td></td>
<td>( \alpha &lt; 1 ) or ( \beta &lt; 1 )</td>
<td>2-S, 1-N</td>
<td>1-S, 3-0</td>
</tr>
<tr>
<td></td>
<td>( \alpha &gt; 1 ) or ( \beta &gt; 1 )</td>
<td>1-SN, 1-N</td>
<td>2-S, 2-0</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 0, 0 &lt; \beta &lt; 1 )</td>
<td>2-N, 1-S</td>
<td>2-S, 2-0</td>
</tr>
<tr>
<td></td>
<td>( \alpha &lt; 1 ) or ( \beta &lt; 1 )</td>
<td>3-N</td>
<td>3-S, 1-0</td>
</tr>
</tbody>
</table>

Table 3: Critical points of (2) with \( f_2(x) \) given in III

<table>
<thead>
<tr>
<th>( f_2(x) )</th>
<th>Critical Points at Infinity</th>
<th>Finite r.p. in the 4 r.p. Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>III. A</td>
<td>1-N</td>
<td>2-S, 2-0</td>
</tr>
<tr>
<td>B</td>
<td>1-N</td>
<td>2-S, 2-0</td>
</tr>
<tr>
<td>C</td>
<td>1-S</td>
<td>1-S, 3-0</td>
</tr>
<tr>
<td>D</td>
<td>1-S</td>
<td>1-S, 3-0</td>
</tr>
<tr>
<td>E</td>
<td>1-S</td>
<td>1-S, 3-0</td>
</tr>
</tbody>
</table>

5-11
Section 6

QUADRATIC OSCILLATORS

This section describes examples of particularly simple systems having stable periodic or almost periodic solutions. Proofs and details are omitted.

1. A Quadratic Relaxation Oscillator in \( x^2 \)

The system

\[
\begin{align*}
\dot{x}_1 &= x_2(1+x_2) \\
\dot{x}_2 &= -x_1(1+x_2)+c(x_2)x_2,
\end{align*}
\]

where \( 0 < c < 2 \) and \( 0 < \alpha < 1 \), is bounded, since it belongs to the class \( D \) of Section 4 with \( a_{11} = 0, a_{12} = -a_{22} = 1, a_{22} = \alpha c \) and \( a_{22} + a_{21} = c(\alpha-1) \leq 0 \). It is evident that (1) has common linear factor \( (1+x_2) \) if either \( c = 0 \) or \( \alpha = 1 \). If \( c = 0 \), the origin is a center, and we have seen in Section 4 that (1) is then the essentially unique bounded quadratic system having a center (Fig. 8, Section 4). If \( c > 0 \) and \( \alpha = 1 \), the trajectories of (1) are the rest points on \( x_2 = -1 \) and appropriately oriented spiral arcs covered by the trajectories of the linear system

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + ax_2,
\end{align*}
\]

which results from omission of the factor \( (1+x_2) \); cf. Fig. 8, Section 4.
We note also that the origin is the unique rest point of (1) if \( \alpha < 1 \) and \( c > 0 \); for \( \alpha = 0 \) the origin is a weak focus which is not a center for (1) unless \( c = 0 \); for \( 0 < \alpha \leq 1, 0 < c < 2 \) the origin is an unstable focus.

**Theorem 1:** If \( 0 < \alpha < 1 \) and \( 0 < c < 3 \), the system (1) possesses a unique limit cycle, \( \Gamma \), which is the \( \omega \)-limit set of every trajectory except the origin.

For fixed \( \alpha (0,1) \):

As \( \alpha \to 0 \), \( \Gamma \) shrinks to the origin and for \( \alpha = 0 \) the weak focus at the origin is asymptotically stable in the large.

As \( \alpha \to 1 \), \( \Gamma \) expands to a limiting "snail curve" of the following description: From the initial point \( P_1 (-c, -1) \) follow the trajectory of (2) to its next intersection, \( P_2 \), with the line \( x_2 = -1 \); then follow the line segment \( P_2 P_1 \).

The variation of \( \Gamma \) with \( \alpha \) is strictly monotone, in the sense of set inclusion of the interior region bounded by \( \Gamma \).

For fixed \( c (0,2) \):

As \( c \to 0 \), \( \Gamma \) approaches the circle with center at the origin and radius \( (2\alpha - \frac{5}{2} + \frac{3}{4} - 2\alpha)^{\frac{1}{2}} \).

As \( c \to 2 \), \( \Gamma \) expands toward infinity. (When \( c > 2 \), the homogeneous quadratic system associated with (1) has a ray solution and (1) has points of finite escape time by Theorem 2, Section 3.)

6-2
The variation of $\Gamma$ is monotone, in the sense mentioned above.

Each point of the line $x_2 = -1$ is a point of ingress for the half-plane $x_2 > -1$, which is therefore positively invariant. Each point of the half-line $x_2 = -\alpha$, $x_1 > 0$ is a point of ingress for the strip 

$-1 < x_2 < -\alpha$, $x_1 > 0$.

If $\Gamma$ is decomposed into a "shell", $\Gamma_s$ and "foot", $\Gamma_f$ defined by $\Gamma_s = \{x | x \in \Gamma, x_2 > -\alpha\} = \Gamma - \Gamma_f$, and if the period, $T$, of (1) on $\Gamma$ is decomposed into transit times over $\Gamma_s$ and $\Gamma_f$, $T = T_s + T_f$, then $T_s/T_f \to 0$ as $\alpha \to 1$ for fixed $\epsilon(0,2)$ (i.e., for $\alpha$ small (1) displays "relaxation" oscillations). The phase-portrait of (1) in $x_2 > -1$ coincides with that of

\begin{align}
  x_1' &= x_2 \\
  x_2' &= -x_1 + c \frac{\alpha x_2}{1 + x_2} x_2
\end{align}

for $\alpha \in [0,1)$, $c \in [0,2)$. System (3) is equivalent to the Liénard equation

\begin{align}
  x_2'' + \left(\frac{x_2}{x_2 + \alpha x_2 + \alpha} + \frac{x_2}{x_2 + \alpha x_2}\right) x_2' + x_2 = 0.
\end{align}

Comment on Proof: The stability of the origin in the case $\alpha = 0$ is obtained by comparison of (3) with the (unbounded) system

\begin{align}
  x_1' &= x_2 \\
  x_2' &= -x_1 + c x_2^2
\end{align}

which has a center at the origin. The limit cycle discussion follows the ideas of [1] and [2].
2. A Quadratic Oscillator in $\mathbb{R}^3$

Theorem 2: In $\mathbb{R}^2$, the only bounded systems of type D, Section 1, having phase-portraits symmetric with respect to the $x_1$-axis are the HT systems

$$\begin{align*}
\dot{x}_1 &= a_{11}x_1 + x_2^2, \\
\dot{x}_2 &= a_{22}x_2 - x_1x_2
\end{align*}$$

(5)

where $a_{11} \leq 0$. Excluding the degenerate case $a_{11} = 0$, the origin is an isolated rest point and is asymptotically stable in the large if $a_{11}a_{22} < 0$. If $a_{11}a_{22} < 0$, the origin is a saddle (which is the ω-limit set of the two other trajectories on the $x_1$-axis) and there are stable rest points at $P_{\pm}(a_{22}, \pm(-a_{11}a_{22})^\frac{1}{2})$. In $x_2 > 0$ the Lyapunov function

$$W(x_1, x_2) = (x_2^2 + a_{11}a_{22}) + (a_{22} - x_1)^2 + a_{11}a_{22} \log \left(\frac{x_2}{-a_{11}a_{22}}\right)$$

(6)

suffices to prove that all trajectories have $P_+$ as ω-limit set; i.e., there is no limit cycle. The quadratic system in $\mathbb{R}^3$

$$\begin{align*}
\dot{x}_1 &= \alpha x_1 - x_2 - x_1x_3 \\
\dot{x}_2 &= x_1 + \alpha x_2 - x_1x_3 \\
\dot{x}_3 &= \beta x_2 + x_1^2 + x_2^2
\end{align*}$$

(7)

where, $\alpha > 0$, $\beta < 0$, reduces, in cylindrical coordinates $\rho = (x_1^2 + x_2^2)^{\frac{1}{2}}$, $\theta = \tan^{-1} \frac{x_2}{x_1}$, and $z = x_3$ to

$$\begin{align*}
\dot{\rho} &= \beta \rho^2 + \rho^2 \\
\dot{\theta} &= \alpha \rho - z_0
\end{align*}$$

(8)
\[ \dot{z} = 1 \] (9.3)

in which \((8)_1, (8)_2\), are identical, except for notation, with the restriction to \(x_2 > 0\) of \((5)_1, (5)_2\) with \(a_{11} = \beta, a_{22} = \alpha\). Every solution of \((7)\) with \(x_1^2 + x_2^2 \neq 0\) therefore approaches the unique circular limit cycle determined by \(z = \alpha, \rho = (\alpha \beta)^{1/2}\).

**Remark 1**: The system \((5)\) was discussed by Birkhoff [5] as a simple model of turbulence and was later made the basis of a more complicated model by Hopf [4], who proved convergence to the rest point in \(x_2 > 0\) by a special argument, here simplified by use of \((6)\). Equation \((7)\) is a "symmetrical" cousin of a quadratic system studied in detail by Sherman [5].

3. A Pseudo-Quadratic System in \(\mathbb{R}^2\) With Invariant Limit Torus

If a limit cycle had been found around the rest point in \(x_2 > 0\) of \((5)\), the rotation device of Theorem 2 would have yielded a limit torus for \((7)\). Since no such limit cycle occurs in a bounded plane system with the appropriate symmetry, we exhibit here an alternative example.

**Theorem 3**: A stable limit cycle bifurcates from the rest point in \(x_2 > 0\) of the system

\[ \begin{align*}
\dot{x}_1 &= -x_1 + x_2^2 \quad \text{(9.1)} \\
\dot{x}_2 &= x_2 - x_1x_2 + cx_2^2 \quad \text{(9.2)}
\end{align*} \]

as \(c\) increases through the value \(\frac{1}{2\sqrt{2}}\). The "pseudo-quadratic" system obtained by replacing \(cx_2^2\) by \(cx_2|x_2|\) in \((9)_2\) has a phase-portrait symmetrical in the \(x_1\)-axis and coincident in \(x_2 \geq 0\) with the phase-portrait of \((9)\). Thus, for \(\frac{1}{2\sqrt{2}} < c < 2\), the following bounded system has a stable limit torus if \(\omega \neq 0\):

\[ \begin{align*}
\dot{x}_1 &= -x_1 + x_2^2 \\
\dot{x}_2 &= x_2 - x_1x_2 + cx_2^2
\end{align*} \]
\begin{align}
\dot{x}_1 &= x_2 - ax_2 \cdot x_1 x_3 + ax_3 (x_1^2 + x_2^2)^{1/2} \\
\dot{x}_2 &= ax_1 + x_2 - x_2 x_3 + ax_3 (x_1^2 + x_2^2)^{1/2} \\
\dot{x}_3 &= -x_2 + x_1^2 + x_2^2
\end{align}

since this system reduces to

\begin{align}
\dot{z} &= -z + \rho^2 \\
\dot{\rho} &= \rho - z \rho + c^2 \\
\dot{\theta} &= \omega
\end{align}

in the cylindrical coordinates defined in Theorem 2.

Section 6 References


QUADRATIC DIFFERENTIAL SYSTEMS

This report is a survey, in preliminary form, of research results obtained by the authors with the sponsorship of the Air Force Office of Scientific Research (AFOSR), Under Contract Number AF 49(638)-1685, entitled "A Study of Quadratically Nonlinear Differential Systems.

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