On eigenvalue problems of real symmetric tensors

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\textbf{A B S T R A C T}

We use variational methods to give a positive answer to a conjecture posed by Liqun Qi [L. Qi, Eigenvalues of a real supersymmetric tensor, J. Symbolic Comput. 40 (2005) 1302–1324] regarding the real eigenvalues of certain higher order tensors.

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1. Introduction

In his pioneer work, Qi \cite{4} introduced the notion of eigenvalues of higher order tensors, and consequently, he studied the existence of both complex and real eigenvalue and eigenvectors. It has now become an important direction in a new branch of numerical multilinear algebra, and it has a wide range of practical applications, for more references, see Qi, Sun, and Wang \cite{5}.

For the complex eigenvalue problem, Qi defined the symmetric hyper-determinant via the resultant for systems of homogeneous polynomials, by which he extended the notion of characteristic polynomials for matrices to a special class of higher order tensors, which he called super-symmetric. The roots of the characteristic polynomials are then the eigenvalues for these tensors.

For the real eigenvalue problem when the order of the super-symmetric tensor is even, Qi further introduced two kinds of eigenvalues, the H-eigenvalues and the Z-eigenvalues. With the aid of variational characterization, he proved (independently by Lim \cite{7}) the existence of the maximum and minimum eigenvalues. These are natural extensions of the counterpart for symmetric matrices. Since it is also well known, a real symmetric $n \times n$ matrix has exactly $n$ real eigenvalues, counting multiplicity, with $n$ linearly independent eigenvectors, Qi further conjectured the same conclusion holds for H-eigenvalues of an even order $n$-dimensional super-symmetric tensor.

A main purpose of this paper is to give a positive answer to this conjecture. We follow the variational approach, and regard the eigenvalues and eigenvectors as critical values and critical points of the associated function confined on a given hypersurface in Euclidean space. Since we are concerned with those saddle points, the critical point theory, in particular, the Ljusternik–Schnirelmann Multiplicity Theorem is employed.

The paper is divided into three parts. The first part is on complex eigenvalues. We slightly extend the notion of eigenvalues by defining the eigenvalues of a tensor $\mathcal{A}$ relative to a given tensor $\mathcal{B}$. By doing so, we will unify the definitions of H-eigenvalues and Z-eigenvalues in Qi \cite{4}, and that of the D-eigenvalues in Qi et al. \cite{6}. It also simplifies and clarifies some assumptions in the discussion of \cite{4}. The second part contains the main result of this paper. First, we define the notion of a weakly symmetric tensor by a group of equality constraints. This is more general than super-symmetry. Next, we prove an

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even order weakly symmetric $n$-dimensional tensor has at least $n$ real eigenvalues with $n$ distinct pairs of real eigenvectors for this generalized eigenvalue problem. This is Theorem 3.6. In the last part, we explore certain assumptions made in our main theorem. For example, the even order assumption on the tensor is crucial. We construct an example for $m = 3$ and $n = 2$ to show the unsolvability of the corresponding real eigenvalue problem. We then turn our attention to study the positive definiteness of $B$. Examples of matrices are provided to investigate the solvability and insolubility of the eigenvalue problem (2.3) with a given positive semi-definite matrix $B$. Subsequently, we are led to pose an additional condition on $A$ and $B$ to ensure the existence of the required multiplicity of real eigenvalues. This is Theorem 4.1.

2. The eigenvalue problem

Let $\mathbb{R}$ be the real field, we consider an $m$-order $n$-dimensional tensor $A$ consisting of $n^m$ entries in $\mathbb{R}$:

$$A = (A_{i_1 \cdots i_m}), \quad A_{i_1 \cdots i_m} \in \mathbb{R}, \quad 1 \leq i_1, \ldots, i_m \leq n.$$  \hfill (2.1)

To an $n$-vector $x = (x_1, \ldots, x_n)$, real or complex, we define an $n$-vector:

$$Ax^{n-1} := \left( \sum_{i_2, \ldots, i_m=1} A_{i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \right)_{1 \leq i \leq n}. \hfill (2.2)$$

**Definition 2.1.** Let $A$ and $B$ be two $m$-order $n$-dimensional tensors on $\mathbb{R}$. Assume that both $Ax^{n-1}$ and $Bx^{n-1}$ are not identical to zero. We say $(A, x)$ is a root of problem (2.3) with a given positive semi-definite matrix $B$. Assume that both $A$ and $B$ are positive semi-definite, then we have:

$$(A - \lambda B)x^{n-1} = 0,$$  \hfill (2.3)

i.e.

$$\sum_{i_2, \ldots, i_m=1} (A_{i_2 \cdots i_m} - \lambda B_{i_2 \cdots i_m}) x_{i_2} \cdots x_{i_m} = 0, \quad i = 1, 2, \ldots, n, \hfill (2.4)$$

possesses a solution.

$\lambda$ is called a $B$-eigenvalue of $A$, and $x$ is called a $B$ eigenvector of $A$.

On the left-hand side of (2.3), $(A - \lambda B)x^{n-1}$ is in fact a set of $n$ homogeneous polynomials in $n$ variables, denoted by $\{P_i^\lambda (x) \mid 1 \leq i \leq n\}$, of degree $(m - 1)$. In the complex field, to study the solution set of a system of $n$ homogeneous polynomials $(P_1, \ldots, P_n)$, in $n$ variables, the idea of the resultant $\text{Res}(P_1, \ldots, P_n)$ is well defined and introduced, we refer to Cox et al. [2] for detail. Applying to our current problem, $\text{Res}(P_1, \ldots, P_n)$ has the following properties:

1. $\text{Res}(P_1, \ldots, P_n) = 0 \Leftrightarrow \exists (\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ satisfying (2.4).
2. $\text{Res}(P_1, \ldots, P_n)$ is an irreducible polynomial with coefficients $A_{i_2 \cdots i_m} - \lambda B_{i_2 \cdots i_m}$ of degree $(m - 1)n^{m-1}$.
3. The degree of $\lambda$ in $\text{Res}(P_1, \ldots, P_n)$ is at most $n(m - 1)^{m-1}$ with coefficients $\text{Res}(Q_1, \ldots, Q_n)$, where $Q_i(x) = \sum_{i_2, \ldots, i_m=1} B_{i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}$ for $i = 1, \ldots, n$.
4. If $B_{i_2 \cdots i_m} = \sigma_i \delta_{i_1 \cdots i_m}$, then $\text{Res}(Q_1, \ldots, Q_n) = (\sigma_1 \cdots \sigma_n)^{(m-1)^{m-1}}$, where $\delta_{i_1 \cdots i_m}$ is the Kronecker symbol:

$$\delta_{i_1 \cdots i_m} = \begin{cases} 1, & i_1 = \cdots = i_m, \\ 0, & \text{otherwise}. \end{cases}$$

The characteristic polynomial $\psi(\lambda) = \text{Res}(P_1^\lambda, \ldots, P_n^\lambda)$ was introduced by Qi [4]. By definition, $\lambda$ is a $B$-eigenvalue of $A$ if and only if $\lambda$ is a root of $\psi$.

**Remark 2.2.** In order for problem (2.3) to make sense, in Definition 2.1, we assume both $Ax^{n-1}$ and $Bx^{n-1}$ are not identical to zero. It is worth noting there is no symmetric assumption on either $A$ or $B$.

In an $m$-order $n$-dimensional tensor $A$ is called symmetric (which Kofidis and Regalia [3] and Qi [4] called supersymmetric), if

$$A_{i_1 \cdots i_m} = A_{\pi(i_1 \cdots i_m)}, \quad \forall \pi \in \mathcal{S}_m,$$

where $\mathcal{S}_m$ is the permutation group of $m$ indices.
It is called skew-symmetric if
\[ A_{i_1\ldots i_m} = \epsilon(\pi) A_{\pi(i_1\ldots i_m)}, \quad \forall \pi \in S_m, \]
where \( \epsilon(\pi) \) is the sign of \( \pi \). For any skew-symmetric \( A \), and for \( m \geq 2 \), we have \( Ax^{m-1} = 0, \forall x \in \mathbb{C}^n \).

**Remark 2.3.** If \( B = \mathcal{I} \), the unit tensor \( \mathcal{I} = (\delta_{i_1\ldots i_m}) \), then the \( B \)-eigenvalues are the eigenvalues, and the real \( B \)-eigenvalues with real eigenvectors are the \( H \)-eigenvalues, in the terminology of [4,5].

Let \( m = 2\ell \) be even and let \( I_2 \) be the \( n \times n \) unit matrix. If \( B = I_2^\ell \), the tensor product of \( \ell \) copies of the unit matrices \( I_2 \), then the \( B \)-eigenvalues are the \( E \)-eigenvalues, and the real \( B \)-eigenvalues with real eigenvectors are the \( Z \)-eigenvalues, in the terminology of [4,5].

Let \( m = 2\ell \) be even and let \( D \) be a symmetric \( n \times n \) matrix. If \( B = D^\ell \), the tensor product of \( \ell \) copies of the matrices \( D \), then the real \( B \)-eigenvalues with real eigenvectors are the \( D \)-eigenvalues, as defined in [6].

**Remark 2.4.** In the case \( m = 2 \), Definition 2.1 coincides with the usual definition of eigenvalues and eigenvectors for matrices.

**Remark 2.5.** The condition \( \text{Res}(Q_1,\ldots,Q_n) \neq 0 \) is sufficient for the existence of eigenvalues of problem (2.3), but not necessary, e.g. see Theorem 3.6 with \( B = I_2^\ell \). Although \( \text{Res}(Q_1,\ldots,Q_n) = 0 \), there exist \( n \) real eigenvalues. We also present in Section 4 an example showing, when \( \text{Res}(Q_1,\ldots,Q_n) = 0 \), (2.3) has no solution for the choices:

\[
A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

3. Real eigenvalues

In this section, we study real eigenvalues with real eigenvectors. Since the resultant theory does not apply, we turn to variational method. Given a real \( m \)-order tensor \( A \), it can be reduced to a homogeneous polynomial of degree \( m \) in \( n \) variables by

\[
f_A(x) := Ax^m = \sum_{i_1\ldots i_m=1}^n A_{i_1\ldots i_m} x_{i_1} \cdots x_{i_m},
\]
where \( x = (x_1,\ldots,x_n) \in \mathbb{R}^n \).

Let \( x^m := x_{j_1}^{f_{j_1}^1} \cdots x_{j_r}^{f_{j_r}^r} \) be a monomial of degree \( m \) where \( 1 \leq j_1 < \cdots < j_r \leq n \), and let \( |\alpha| = \alpha_1 + \cdots + \alpha_r = m \). We have:

\[
f_A(x) = \sum_{|\alpha|=m, 1 \leq j_1 < \cdots < j_r \leq n} \left( \sum_{(i_1,\ldots,i_m) \sim (j_1^{f_{j_1}^1},\ldots,j_r^{f_{j_r}^r})} A_{i_1\ldots i_m} x_{j_1}^{f_{j_1}^1} \cdots x_{j_r}^{f_{j_r}^r} \right).
\]

where \( (j_1^{f_{j_1}^1},\ldots,j_r^{f_{j_r}^r}) \) means \( j_v \) is repeated for \( \alpha_v \) times for \( 1 \leq v \leq r \), and \( (i_1,\ldots,i_m) \sim (i_1',\ldots,i_m') \) means there exists a \( \pi \in S_m \) such that \( \pi(i_1,\ldots,i_m) = (i_1',\ldots,i_m') \).

It is easily seen \( f_A \) has at most \( \frac{(m+n-1)!}{m!(n-1)!} \) terms. Since \( f_A \) is a homogeneous polynomial of degree \( m \), it is an \( m \)-homogeneous function:

\[
f_A(\lambda x) = \lambda^m f_A(x), \quad \forall \lambda \in \mathbb{R}.
\]

A straightforward calculation shows:

\[
(x, \nabla f_A(x)) = mf_A(x),
\]
where \( (\cdot,\cdot) \) denotes the standard inner product on \( \mathbb{R}^n \), and \( \nabla f \) denotes the gradient of \( f \).

**Definition 3.1.** An \( m \)-order \( n \)-dimensional real tensor \( A \) is called positive definite, resp. positive semi-definite, if \( f_A(x) = Ax^m > 0 \), resp. \( f_A(x) = Ax^m \geq 0 \), \( \forall x \in \mathbb{R}^n \setminus \{0\} \).

From the \( m \)-homogeneity, we see if \( A \) is positive definite, then necessarily \( m \) is even.

**Example 3.2.** The \( m = 2\ell \)-order tensors \( A_1 = \mathcal{I} \), \( A_2 = I_2^\ell \), and \( A_3 = D_2^\ell \), where \( D_2 \) is a positive definite real \( n \times n \) matrix, are all positive definite, since \( A_1 x^m = \sum_{i=1}^m x_i^m \), \( A_2 x^{2\ell} = (\sum_{i=1}^m x_i^2)^\ell \), and \( A_3 x^{2\ell} = (Dx,x)^\ell \).
Definition 3.3. An $m$-order $n$-dimensional real tensor $A$ is called weakly symmetric, if
\[ \nabla f_A(x) = mAx^{m-1}, \quad \forall x \in \mathbb{R}^n \] (3.2)
and the right-hand side is not identical to zero.

If $A$ is not symmetric, we can define its symmetrization $\tilde{A} = (\tilde{A}_{i_1\cdots i_m})$ by
\[ \tilde{A}_{i_1\cdots i_m} := \frac{1}{m!} \sum_{\pi \in S_m} A_{\pi(i_1\cdots i_m)}. \]

It follows that
1. $A$ and $\tilde{A}$ share the same homogeneous function, i.e. $f_A(x) = f_{\tilde{A}}(x)$.
2. If $A$ is (super)-symmetric, then it is weakly symmetric. In fact, if $A$ is (super)-symmetric, then by (3.1), we have:
\[ f_A(x) = m! \sum_{|\alpha|=m, 1 \leq j_1 < \cdots < j_r \leq n} \frac{m!}{\alpha_1! \cdots \alpha_r!} A_{j_1^{\alpha_1} \cdots j_r^{\alpha_r} x_1^{\alpha_1} \cdots x_r^{\alpha_r}}, \]
(3.3)
and
\[ \nabla f_A(x) = m \sum_{|\beta|=m-1, 1 \leq k_1 < \cdots < k_r \leq n} \frac{(m-1)!}{\beta_1! \cdots \beta_r!} A_{k_1^{\beta_1} \cdots k_r^{\beta_r} x_1^{\beta_1} \cdots x_r^{\beta_r}} \]

hence, $\nabla f_A(x) = mAx^{m-1}$.

Remark 3.4. There exist weakly symmetric tensors which are not (super)-symmetric. For example, for $m = 4$ and $n = 2$, let $I_2 = (\delta_{i_1j_1}, \delta_{i_2j_2})$, i.e.
\[ A_{i_1i_2j_1j_2} = \begin{cases} 1, & (i_1i_2j_1j_2) = (1111), (1122), (2211), (2222), \\ 0, & otherwise. \end{cases} \]

It is weakly symmetric, since $f_A(x) = (x_1^2 + x_2^2)^2$ and
\[ \nabla f_A(x) = 2(x_1^2 + x_2^2)(x_1, x_2) = 2Ax^3. \]

However, the symmetrization $\tilde{A}$ of $A$ reads as follows:
\[ \tilde{A}_{i_1i_2j_1j_2} = \begin{cases} 1, & (i_1i_2j_1j_2) = (1111), (2222), \\ \frac{1}{2}, & (i_1i_2j_1j_2) = (1122), (1212), (1212), (2121), (2112), (2211), \\ 0, & otherwise. \end{cases} \]

Remark 3.5. An $m$-order $n$-dimensional real tensor $A$ has $n^m$ independent entries. A (super)-symmetric $m$-order $n$-dimensional real tensor has only $\binom{m+n-1}{n-1}$ independent entries, which are the coefficients of the monomials in $f_A(x)$, see (3.3). In contrast, a weakly symmetric tensor has at least $n^m - n|m+n-2,n-1| + n$ independent entries, because the system (3.2) consists of $(m-1)$-order homogeneous polynomials, and each equation contains $\binom{m+n-2}{n-1} - 1$ constraints of the coefficients of these polynomials. However, these constraints may not be independent, e.g. for $m = 2$, in linear systems, only half of the $n(n-1)$ constraints are independent.

Now we assume both $A$ and $B$ are weakly symmetric real tensors of the same order $m$ and of the same dimension $n$. Moreover, we assume $B$ is positive definite, which implies $m$ is even. We study the eigenvalue problem (2.3) $(A - \lambda^2B)x^{m-1} = 0$. Our main result relies on the Ljusternik-Schnirelmann Multiplicity Theorem in critical point theory, see e.g. [1, 8]. It asserts a $C^1$-function $f$ defined on a $C^1$-compact manifold $M$ has at least $\text{cat}(M)$ critical points (counting multiplicity), where $\text{cat}(M)$, the Ljusternik-Schnirelmann category of $M$, is a numerical topological invariant of $M$, i.e. $\text{cat}(M)$ is completely determined by the homotopy type of $M$.

Theorem 3.6. Assume $A$ is a weakly symmetric tensor and $B$ is a weakly symmetric positive definite tensor, both have the same order $m$ and dimension $n$. Then problem (2.3) has at least $n$ real eigenvalues, counting multiplicity, with $n$ distinct pairs of real eigenvectors.
Proof. 1° We let $N := \{ x \in \mathbb{R}^n \mid g(x) = Bx^m = 1 \}$, and we claim $N$ is an $(n - 1)$-dimensional smooth manifold in $\mathbb{R}^n$. In fact, from
\[
(x, \nabla g(x)) = mg(x) = m \quad \text{on} \quad N,
\]
we see $\nabla g(x) \neq 0, \forall x \in N$, this implies $N = g^{-1}(1)$ is an $(n - 1)$-dimensional smooth manifold in $\mathbb{R}^n$.

2° $N$ is homeomorphic (in fact diffeomorphic) to $S^{n-1}$, the unit sphere in $\mathbb{R}^n$. We construct this diffeomorphism as follows. Since $S^{n-1}$ is compact, and $B$ is positive definite, there exists a constant $c > 0$ such that $g(x) \geq c > 0, \forall x \in S^{n-1}$. Given $x \in S^{n-1}$, let
\[
\lambda = \lambda(x) = \frac{1}{g(x)^{1/m}},
\]
we then have $g(\lambda x) = \lambda^m g(x) = 1$. Define the map $\phi : S^{n-1} \to N$ by $x \mapsto \lambda(x) \cdot x$, it is then a standard exercise to see $\phi$ is indeed a diffeomorphism. This implies $N$ is compact.

3° Noting both $f_A$ and $g$ are even functions, they are invariant under the $\mathbb{Z}_2$-action. Here $\mathbb{Z}_2$ denotes the two element group consisting of the identity map and the antipodal map on $\mathbb{R}^n \setminus \{0\}$. Let $M = N/\mathbb{Z}_2$, then $M$ is again a compact smooth manifold of dimension $n - 1$, homeomorphic to $\mathbb{R}^{n-1} / \mathbb{Z}_2$.

4° Since the $L$–$S$ category is a topological invariant,
\[
\text{cat}(M) = \text{cat}(\mathbb{R}^{n-1}) = n.
\]
We now apply the Ljusternik–Schnirelmann Multiplicity Theorem directly, and conclude there exist at least $n$ distinct critical points of $f_A(x)$ on $M$. Furthermore, at each critical point $x_0 \in M$, there exists a Lagrangian multiplier $\lambda_0 \in \mathbb{R}$ such that
\[
\nabla f_A(x_0) - \lambda_0 \nabla g(x_0) = 0.
\]
Since $g = f_B$, and both $A$ and $B$ are weakly symmetric, it follows that
\[
(A - \lambda_0 B)x_0^{m-1} = 0.
\]
In this case,
\[
\lambda_0 = \frac{f_A(x_0)}{f_B(x_0)} = f_A(x_0),
\]
and $(\lambda_0, x_0)$ is a solution of (2.3). We thus have established the existence of $n$ real eigenvalues and $n$ distinct pairs of eigenvectors $\{ (\lambda_i, x_i) \}_{i=1}^n$. \qed

Corollary 3.7. If $A$ is an even order real (super)-symmetric tensor of dimension $n$, then there exist at least $n$ $H$-eigenvalues (and $Z$-eigenvalues, and $D$-eigenvalues) with $n$ distinct pairs of eigenvectors.

Remark 3.8. Among the $n$ critical points of $f_A$ on $N = f_B^{-1}(1)$, there are a maximizer and a minimizer, whose existence has been proved by Qi [4] and Lim [7]. The main contribution of this paper is to fill up the intermediate eigenvalues, and then to answer Conjecture 3 in [4] positively.

4. Further discussions

We end this paper with a few questions and answers.

1. Why is $m$ even?

In Theorem 3.6, it is assumed the order $m$ is even. Without this assumption, one cannot conclude the existence of any real eigenvalues in general. Consider the following example where $m = 3$ and $n = 2$. We choose
\[
A = (A_{ij1213}), \quad 1 \leq i_1, i_2, i_3 \leq 2,
\]
where $A_{111} = b, A_{222} = d, -A_{112} = -A_{121} = a = A_{221} = A_{212} = A_{122} > 0$, and
\[
B = (B_{ij1213}), \quad 1 \leq i_1, i_2, i_3 \leq 2,
\]
where $B_{111} = B_{222} = 1, B_{112} = B_{121} = B_{221} = B_{222} = B_{212} = B_{122} = 0$. Then the eigenvalue problem (2.3) reduces to solving the following system:
\[
(b - \lambda)x^2 - 2axy + ay^2 = 0, \\
-ax^2 + 2axy + (d - \lambda)y^2 = 0.
\]
Setting
\[
c = \frac{b - d}{2}, \quad \xi = \frac{b + d}{2} - \lambda,
\]

the resultant of (4.1) becomes

\[
p(\lambda) = \det \begin{pmatrix}
\xi + c & 0 & -a & 0 \\
-2a & \xi + c & 2a & -a \\
a & -2a & \xi - c & 2a \\
0 & a & 0 & \xi - c
\end{pmatrix} = \xi^4 - (2\xi^2 - 6\xi^2)\xi^2 + [c^4 - 6a^2c^2 + 8a^3c - 3a^4].
\]

In particular, if \(c = \sqrt{3}a\), then \(p(\lambda) = \xi^4 + (8\sqrt{3} - 12)a^4 > 0\). Thus, the system (4.1) has no real solution.

2. What happens if \(B\) is only positive semi-definite?

We consider the special case \(B = (B_{i_1\ldots i_m})\), for \(m\) even and

\[
B_{i_1\ldots i_m} = \begin{cases}
\sigma_i, & (i_1\ldots i_m) = (i\ldots i) \text{ for } 1 \leq i \leq r < n, \\
0, & \text{otherwise},
\end{cases}
\]

where \(\sigma_1 \geq \cdots \geq \sigma_r > 0\). Consequently, \(Bx = \sum_{i=1}^r \sigma_i x_i^n\), and \(B\) is positive semi-definite but not positive definite. One may wonder if problem (2.3) has at least \(r\) real eigenvalues without further assumptions on the real symmetric tensor \(A\). In fact, even when \(m = 2\), i.e. when \(A\) is a symmetric matrix, we know, by considering the following example, the conclusion is not true in general:

\[
A = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix} \text{ and } B = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Since

\[
\det(A - \lambda B) = \det \begin{pmatrix}
-\lambda & 0 & 1 & 0 \\
0 & -\lambda & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix} = 1,
\]

there is no solution.

However, for matrices, we subsequently search for a sufficient condition for which (2.3) has at least \(r\) real eigenvalues.

Let \(\mathbb{R}^n = X = Y \oplus Z\), where

\[
Y = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_{r+1} = \cdots = x_n = 0\},
\]

\[
Z = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = \cdots = x_r = 0\}.
\]

\(\forall x \in X\), we have the direct sum decomposition \(x = y + z\) for \(y \in Y\) and \(z \in Z\). We write

\[
A = \begin{pmatrix}
C_1 & F \\
F^\perp & C_2
\end{pmatrix} \text{ and } B = \begin{pmatrix}
D & 0 \\
0 & 0
\end{pmatrix},
\]

where \(C_1, C_2, F\) are \(r \times r\), \((n-r) \times (n-r)\), \((n-r) \times r\) matrices resp., and \(D = \text{diag}(\sigma_1, \ldots, \sigma_r)\) for \(\sigma_1 \geq \cdots \geq \sigma_r > 0\).

The eigenvalue problem (2.3) now becomes the following system of linear equations:

\[
(C_1 - \lambda D)y + Fz = 0,
\]

\[
F^\perp y + C_2z = 0.
\]

If \(C_2\) is an invertible \((n-r) \times (n-r)\) matrix, then \(z = -C_2^{-1}F^\perp y\), and (4.2) is reduced to

\[
(C_1 - FC_2^{-1}F^\perp - \lambda D)y = 0.
\]

Since \(C_1 - FC_2^{-1}F^\perp\) is symmetric, there are \(r\) real eigenvalues (counting multiplicity).

In light of the above discussion, we now generalize this observation to the higher order tensor setting:

**Theorem 4.1.** Assume \(m\) is even and \(0 < r < n\). Assume the \(m\)-order \((n-r)\)-dimensional sub-tensor \(\tilde{A} = (\tilde{A}_{i_1\ldots i_m})\) of an \(n\)-dimensional tensor \(A\), i.e.

\[
\tilde{A}_{i_1\ldots i_m} = A_{i_1\ldots i_m} \text{ for } r+1 \leq i_1, \ldots, i_m \leq n
\]

is positive definite on \(Z\), i.e. \(\tilde{A}^{m:0} > 0\), \(\forall z \in Z \setminus \{0\}\). If the tensor \(B = (B_{i_1\ldots i_m})\) satisfies \(B_{i_1\ldots i_m} = 0\) for any one of the indices \((i_1\ldots i_m)\) greater than \(r\), and further, if the \(m\)-order \(r\)-dimensional sub-tensor \(\tilde{B}\) is positive definite on the \(r\)-dimensional space \(Y\), then (2.3) has at least \(r\) real eigenvalues with \(r\) distinct pairs of eigenvectors.
Proof. By the assumption of $\mathcal{B}$, we have $Bx^m = B\hat{y}^m = \hat{B}y^m$. Similar to the proof of Theorem 3.6, the constraint $g(x) = f_\mathcal{B}(x) = Bx^m = \hat{B}y^m = 1$ defines a smooth manifold $N = g(1)$. Since $\hat{B}$ is positive definite on $Y$, by the same proof of Theorem 3.6, $N$ is diffeomorphic to the cylinder $S^{m-1} \times \mathbb{R}^{m-1}$. This is, however, a noncompact manifold. In order to apply the Ljusternik–Schnirelmann theory, we have to control the behavior of the function $f_\mathcal{A}(x)$ on $N$ at infinity.

From the positivity of $\hat{A}$, we have a constant $c_0 > 0$ such that

$$\hat{A}z^m \geq c_0 \|z\|^m,$$

where $\|z\| = (\sum_{i=1}^m z_i^2)^{1/2}$.

Since now $\hat{B}y^m = 1$, and $\hat{B}$ is positive definite, $\|y\| = (\sum_{i=1}^m y_i^2)^{1/2}$ is bounded, therefore, we have a constant $C_1 > 0$ such that

$$|f_\mathcal{A}(x) - \hat{A}z^m| \leq C_1 (\|z\|^{m-1} + 1).$$

Combining it with (4.4), it follows

$$f_\mathcal{A}(x) \geq c_0 \|z\|^m - C_1 \|z\|^{m-1} - C_1.$$

Thus, the function $f_\mathcal{A}(x) = \hat{A}z^m$ is bounded from below but not bounded from above on $N$. Again, the Ljusternik–Schnirelmann Multiplicity Theorem can be applied to this case, if the Palais–Smale condition holds. The Palais–Smale condition for $f_\mathcal{A}$ on $N$ means: any sequence $\{u^k\}_{k=1}^\infty \subset N$, along which

$$f_\mathcal{A}(u^k) \to c$$

for some $c \in \mathbb{R}$, and

$$df_\mathcal{A}(u^k) = m \left[ A(u^k)^{m-1} - \frac{(A(u^k)^{m-1}) B(u^k)^{m-1}}{\|B(u^k)^{m-1}\|^2} B(u^k)^{m-1} \right] \to 0,$$

contains a convergent subsequence (where $df_\mathcal{A}(u)$ is the projection of $\nabla f_\mathcal{A}(u)$ onto $T_u(N)$, the tangent space of $N$ at $u$).

We write $u^k = y^k + z^k$. Since $\{u^k\} \subset N$, $\{y^k\}$ is bounded, hence contains a convergent subsequence $\{y^k_j\}$. Moreover, from (4.5),

$$|f_\mathcal{A}(u^k) - \hat{A}(z^k)| \leq C_1 (\|z^k\|^{m-1} + 1).$$

Again by (4.4), we have

$$c_0 \|z^k\|^m \leq \hat{A}(z^k)^m.$$

Together with the condition $f_\mathcal{A}(u^k) \to c$, it follows that $\{z^k\}$ is bounded, consequently, there exists a subsequence of $k_j$, denoted by $k_j'$, such that $z^{k_j'}$ is convergent. Thus $u^{k_j'} = y^{k_j'} + z^{k_j'}$ is convergent, and the Palais–Smale condition is verified.

Since $f_\mathcal{A}$ and $g$ are both even functions, they descend to define maps on $M = N/\mathbb{Z}_2$, which deformation retracts to $\mathbb{R}^{p-1}$. Our assertion now follows from the fact $\text{cat}(M) = \text{cat}(\mathbb{R}^{p-1}) = r$. □

References