Attractors of partial differential evolution equations and estimates of their dimension

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1983 Russ. Math. Surv. 38 151


View the table of contents for this issue, or go to the journal homepage for more.

Download details:
IP Address: 129.236.21.103
The article was downloaded on 13/01/2011 at 17:04

Please note that terms and conditions apply.
Attractors of partial differential evolution equations and estimates of their dimension

A.V. Babin and M.I. Vishik

CONTENTS

Introduction 151

§ 1. Maximal attractors of semigroups generated by evolution equations 156

§ 2. Examples of parabolic equations and systems having a maximal attractor 158

§ 3. The Hausdorff dimension of invariant sets 164

§ 4. Estimate of the change in volume under the action of shift operators generated by linear evolution equations 167

§ 5. An upper bound for the Hausdorff dimension of attractors of semigroups corresponding to evolution equations 172

§ 6. A lower bound for the dimension of an attractor 175

§ 7. Differentiability of shift operators 177

§ 8. Estimates of the Hausdorff dimension of an attractor of a two-dimensional Navier-Stokes system 183

§ 9. Upper and lower bounds of the dimension of parabolic equations and parabolic systems 194

§ 10. Attractors of semigroups having a global Lyapunov function 202

§ 11. Regular attractors of semigroups having a Lyapunov function 206

References 210

Introduction

One of the main problems in the theory of evolution differential equations

\[ \partial_t u(t) = Au(t), \quad u|_{t=0} = u_0 \quad (\partial_t = \partial/\partial t) \]

is the study of the behaviour of their trajectories \( u(t) = u(t, x) \) as \( t \to +\infty \).

For linear equations (1) the behaviour of trajectories as \( t \to +\infty \) depends on the spectrum of the spatial part of the operator \( A \) in question. For non-linear evolution equations the problem is considerably more complex.

Finite-dimensional dynamical systems have been studied extensively in the mathematical literature (see, for example [1]–[3]). Infinite-dimensional non-linear evolution equations have been studied to a much lesser extent, although in recent years a number of substantial results have been obtained also for such equations (see, for example, [4], [5], [3], [9]–[18], [26], [33]–[47]).
In the study of the behaviour of trajectories of evolution equations as \( t \to +\infty \) attractors play an important role. The semigroup \( \{S_t, t \geq 0\} \) corresponding to (1) is the family of operators \( S_t \) acting on a Banach space \( E \), \( S_t : E \to E \), such that \( S_t u_0 = u(t) \), where \( u(t) \) is the solution of (1). This definition clearly assumes unique solubility of (1). A maximal attractor of \( \{S_t\} \) for (1) is a closed bounded set \( \mathcal{M} \subset E \) with the following properties: 1) \( \mathcal{M} \) is invariant: \( S_t \mathcal{M} = \mathcal{M} \) for all \( t \geq 0 \) and 2) \( \mathcal{M} \) is attracting: for any bounded \( B \subset E \) the distance \( \text{dist}_E (S_t(B), \mathcal{M}) \to 0 \) as \( t \to +\infty \) (see §I). For the sake of brevity we call maximal attractors simply attractors.

To begin with we give an example of an attractor of a particularly simple structure. This is the case of gradient-like equations (1) of the form

\[
\partial_t u = -a(u) \frac{\delta \Phi(u)}{\delta u(x)} , \quad a(u) > 0,
\]

where \( \delta \Phi(u)/\delta u(x) \) is the functional derivative of some functional \( \Phi(u) \). These as well as some other equations possess a global Lyapunov function, that is, a functional \( \Phi(u) \) whose values decrease along the trajectories, with the exception of stationary trajectories on which \( \Phi \) is constant (see §10). Under certain natural conditions the attractors of such equations have the form

\[
\mathcal{M} = \bigcup_{j=1}^\kappa M(z_j),
\]

where \( \{z_j\} \) is the set of stationary solutions of (1) (we assume that there are finitely many, say \( \kappa \), such solutions) and \( M(z_j) \) is the maximal unstable invariant manifold passing through the point \( z_j \). This \( M(z_j) \) consists of all the trajectories \( u(\tau) \) emanating from \( z_j \) for increasing \( \tau \) or, more precisely, \( u(\tau) \to z_j \) as \( \tau \to -\infty \) (see §10). If the semigroup \( \{S_t\} \) corresponding to (1) satisfies certain differentiability conditions, then the \( M(z_j) \) are \( C^1 \)-manifolds in \( E \) of finite dimension \( n(z_j) \). Attractors \( \mathcal{M} \) having the structure (2), where the \( M(z_j) \) are \( C^1 \)-manifolds, are called regular (see §11).

We note that under certain conditions that ensure the existence of a regular attractor \( \mathcal{M} \) for \( \{S_t\} \) each individual trajectory \( u(t) = S_t u_0 \) tends as \( t \to +\infty \) to one of the stationary points \( z_j \). However, the sets \( S_t(B) \), where \( B \) is any bounded set in \( E \) (for example, \( B = B_R \) is a ball of radius \( R \) in \( E \)) tend to the attractor described above. For a linear parabolic equation this corresponds to the fact that each solution \( u(t) \) tends to zero or diverges at infinity as \( t \to +\infty \), and bounded sets approach a linear subspace \( \mathcal{M} \) with a basis of unstable eigenvectors of \( A \) (that is, corresponding to eigenvalues \( \lambda \) with \( \text{Re} \lambda > 0 \)).

Attractors of parabolic equations with a gradient-like right-hand side whose principal part is a monotonic operator have the structure (2) (for details, see §10). Regular attractors of the form (2) occur for gradient-like parabolic equations acting of function spaces of smoothness \( C^{2+\alpha} \), \( \alpha > 0 \), and also for dissipative hyperbolic equations, for example,

\[
\partial_t^2 u + \varepsilon \partial_t u = \Delta u - f(u) - g(x), \quad \varepsilon > 0,
\]
where the non-linear function \( f(u) \) satisfies certain natural conditions (see §11 and [26]).

In the case of finite-dimensional dynamical systems an object similar to (2) was studied by Smale [6]. Palais and Smale [8] studied infinite-dimensional gradient-like dynamical systems in the context of Morse theory [7]. Problems of the qualitative theory of second-order parabolic equations and of systems (1) are treated in Henry’s book [5], in which there is an account of a number of results on gradient flows related to parabolic equations and systems. There is also an extensive list of references.

For evolution equations that do not have a global Lyapunov function the structure of attractors is considerably more complex even for finite-dimensional dynamical systems (see, for example [3] and [21]). Only certain properties of invariant sets and problems connected with the finiteness and upper and lower bounds for the Hausdorff dimension of attractors have so far been discussed for general evolution equations and systems of partial differential equations of the form (1). For example, Ladyzhenskaya [9] proved the existence of an invariant set for a two-dimensional Navier-Stokes system and established that any trajectory of this set can be restored from its finite-dimensional projection. Mallet-Paret [10] proved that the Hausdorff dimension of an invariant set for a certain quasilinear parabolic equation is finite. The fact that the Hausdorff dimension of any invariant set of a two-dimensional Navier-Stokes system is finite was established by Foias and Temam [11]. Il’yashenko [12] was the first to give an upper estimate of the Hausdorff dimension of the attractor \( \mathcal{M} \) of the \( N \)th-order Caloikin approximation for a two-dimensional Navier-Stokes system with periodic boundary conditions: 

\[
\dim \mathcal{M} \leq C \nu^{-1} \cdot \nu^{-2} (|| g || (\log || g || + 2))^{1/2} + || g ||^{-1}.
\]

The Hausdorff dimension of an attractor for a two-dimensional Navier-Stokes system with zero boundary conditions was found to be bounded by an exponential in \( 1/\nu \) (see [15] and [35]). We obtain in §8 a power estimate
To give a lower estimate of the dimension $\dim \mathcal{F}$ of a maximal attractor, we use the fact that for any stationary point $z$ of (1) the local unstable invariant manifold $M(z)$ is contained in $\mathcal{R}$ (see §6). Therefore, $\dim M(z) \leq \dim \mathcal{R}$. The dimension of $M(z)$ is equal to the number of eigenvalues $\mu_j$ of $A'(z)$ located in the half-plane $\Re \mu > 0$ (see §9). On the basis of these arguments we obtain in §8 the following lower bound for an attractor of a two-dimensional Navier-Stokes system in the periodic case indicated above:

$$\dim \mathcal{F} \geq C_1 \alpha_0^{-1} = C_1 \Re.$$  

For this purpose we use the example of an unstable periodic flow constructed by Meshalkin and Sinai [17], which was later generalized by Yudovich [18]. In this paper it is shown that the variational equation for a two-dimensional Navier-Stokes system at a certain explicitly constructed stationary point $z(x) = (z^1(x), z^2(x))$ (see §8) has unstable solutions. It follows easily from the construction (see §8) that the dimension of the space of unstable directions of the variational equation at $z(x)$ is of the order $Ca^z_0$, and this is also the dimension of the unstable invariant manifold $M(z)$ passing through $z$. Now (5) follows from this.\(^{(1)}\)

In §9 we derive upper and lower bounds for the dimension of attractors of quasilinear parabolic equations and systems containing a small parameter $\nu$ at the second-order terms and a term $\frac{\lambda}{\nu}$, where $\lambda$ is a large parameter. For example, the following upper estimate for the dimension of $\mathcal{F}$ can be established under certain conditions:

$$\dim \mathcal{F} \leq C_1 \nu^{-\frac{1}{2}} \lambda^{n/2}.$$  

The same estimate holds for the dimension of an attractor of a linear parabolic equation with a self-adjoint spatial part containing parameters $\nu$ and $\lambda$ as specified above. An estimate similar to (6) of the dimension of an attractor holds for the system of equations of the type of chemical kinetics (see §9). Estimating the number of unstable directions at a stationary point $z$ of such a system we derive in §9 a lower estimate similar to (6):

$$\dim \mathcal{F} \geq C_1 \nu^{-\frac{1}{2}} \lambda^{n/2}.$$  

In §§9 and 11 we show for parabolic equations

$$\partial_t u = \Delta u - f(u) - \lambda u - g(x),$$

where $u|_{\partial \Omega} = 0$ or $u$ satisfies periodic boundary conditions that

$$\dim \mathcal{F} = N(\lambda) + o(N(\lambda)),$$

where $N(\lambda)$ is the number of eigenvalues of $-\Delta$ smaller than $\lambda$.

\(^{(1)}\) We note that the points of $M(z)$ under the action of $S_t$ may give, generally speaking, in the limit $t \to +\infty$ a set whose dimension is considerably lower than that of $M(z)$.  

154  

A.V. Babin and M.I. Vishik
The present paper is devoted to a systematic account of the type of problems outlined above. We consider equations not depending explicitly on time. These equations give rise to operator semigroups $S_t$, $t \in \mathbb{R}$, $t > 0$. Equations of the form $\partial_t u = A(t, u)$, where $A(t, \cdot)$ depends periodically on $t$ with a period $l$, can be treated along the same lines. This equation gives rise to a semigroup of shift operators $S_{kl}$ over times that are multiples of the period. All the main considerations in the periodic case can go parallel to the autonomous case with the exception of the results of § 10 and 11.

We now describe briefly the contents of the paper. In § 1 we prove a general theorem on the existence of an attractor for semigroups. We assume that the semigroup $\{S_t\}$ is uniformly bounded and that $\{S_t\}$ has a compact absorbing set. In § 2 we present examples of parabolic equations and systems having attractors. These include a general quasilinear second-order parabolic equation with coefficients depending on $u$ whose solutions are uniformly bounded in $t$ in the space $C$ and also parabolic equations and systems with a monotonic principal part. In particular, this includes the system of equations of the type of chemical kinetics. In § 8 we show that the two-dimensional Navier-Stokes system also has an attractor.

In § 3, following the papers by Douady and Oesterlé [42] and Il'yashenko [47], we introduce the concept of a $(k, N)$-contracting map $S$ of a set $X$ in $H$. We prove that the Hausdorff dimension of sets $X$ invariant under $S$ is finite. We also introduce the flattening map for which a similar theorem holds.

In § 4 we estimate the changes in volume under the action of shift operators generated by linear evolution equations. The estimates are subsequently used for the variational equations corresponding to the nonlinear equations in question.

In § 5 we define the concept of uniform quasidifferentiability of a map $S$ on $X \subset E$. If an operator $S_t$, $t > 0$, of the semigroup corresponding to (1) is uniformly quasidifferentiable on an attractor $\mathcal{A}$ and the spatial part of the variational equation for (1) along trajectories located in $\mathcal{A}$ satisfies certain inequalities, we derive in § 5 an upper bound for the Hausdorff dimension of the attractor. A lower bound for the dimension of the attractor is based on the inclusion $M(z) \subset \mathcal{A}$, where $M(z)$ is a local unstable invariant manifold of $\{S_t\}$ and $\mathcal{A}$ is the attractor. The properties of the manifold $M(z)$ are investigated in § 6 (see [19]), where we prove this inclusion. The manifold $M(z)$ is of class $C^1$ if the semigroup $\{S_t\}$ is differentiable and its differential satisfies a Hölder condition.

In § 7 we describe a general scheme by which smoothness properties of operators $S_t$ corresponding to equations of the form (1) can be established. We also show that the operators $S_t$ corresponding to the parabolic equations and systems of equations studied in § 2 have the required differential properties in various function spaces, provided that they satisfy certain additional conditions.
In §8 we derive upper and lower estimates for the dimension of an attractor for the two-dimensional Navier-Stokes system. In §9 we give upper and lower bounds for $\dim \mathcal{A}$ in the case of parabolic equations and systems of equations, in particular, for a system of equations of the type of chemical kinetics. §§10 and 11 deal with evolution equations having a global Lyapunov function.

In conclusion we mention that in this paper, as already pointed out, the term attractor is understood to be a maximal bounded invariant attracting set for a given dynamical system. As Arnol'd has remarked, such a set is, in general, wider than the set of physically observable limiting regimes that establish themselves in a dynamical system after a long time interval.

§1. Maximal attractors of semigroups generated by evolution equations

An evolution equation of the form $\partial_t u = A(u), u|_{t=0} = u_0, u(t) \in E$, where $E$ is a Banach space gives rise to a semigroup of operators such that $S_t u_0 = u(t)$ if the given initial problem has a unique solution. In §2 we give examples of such semigroups generated by various partial differential equations. Here we give a general definition of a maximal attractor of a semigroup and prove its existence for semigroups satisfying certain general conditions.

Definition 1.1. Let $E$ be a Banach space, $\{S_t, t \geq 0\}$ a semigroup of operators $S_t : E \to E$, $S_t \circ S_t = S_{t+t}$, and $S_0 = I$ the identity operator. A bounded set $\mathcal{A}, \mathcal{A} \subset E$, that is closed in $E$ is said to be a maximal attractor of $\{S_t\}$ if

1) $\lim_{t \to +\infty} \text{dist}(S_t B, \mathcal{A}) = 0$ for any bounded set $B \subset E$ (condition of attraction), where $\text{dist}(X, Y) = \sup_{x \in X} \inf_{y \in Y} \| x - y \|, \| \cdot \| = \| \cdot \|_E$;

2) $S_t \mathcal{A} = \mathcal{A} \quad \forall \ t \geq 0$ (invariance condition).

Theorem 1.1. Suppose that $\{S_t\}$ satisfies the following conditions:

1) The semigroup $\{S_t\}$ is uniformly bounded, that is, $\forall \ R > 0 \exists C(R)$ such that $\| S_t u \| \leq C(R)$ if $\| u \| \leq R \quad \forall \ t \in [0, +\infty)$.

2) There is a compact absorbing set $B_0$ in $E$, that is, for any bounded set $B \subset E$ there is a number $T$ such that $S_t B \subset B_0$ for $t \geq T$.

3) The operators $S_t : E \to E$ are continuous for $t \geq 0$.

Then the semigroup $\{S_t\}$ has a compact maximal attractor.

Proof. We set

$$\mathcal{A} = \bigcap_{t \geq 0} F_t, \quad F_t = \bigcup_{t' \geq t} S_{t'} B_0.$$  

Since $B_0$ is absorbing, the sets $F_t$ for large $t$ are contained in $B_0$ and are consequently compact. Since $F_0 \subset F_t$ for $\theta > t$, their intersection is non-empty.
It follows from the definition (1.1) of \( \mathcal{A} \) that

\[
1.2 \quad x \in \mathcal{A} \iff \exists \text{ sequences } \{x_j\} \subset B_0, \ \{t_j\} \subset \mathbb{R}_+ \quad \text{ such that } \quad t_j \to +\infty, \ S_{t_j}x_j \to x.
\]

We claim that the set \( \mathcal{A} \) defined by (1.1) has the invariance property. Let us verify that \( S_\theta \mathcal{A} \supseteq \mathcal{A} \) for \( \theta > 0 \). Let \( x \in \mathcal{A} \) and let \( \{x_j\} \) and \( \{t_j\} \) be the sequences defined in (1.2). For large \( j \) we set \( t_j' = t_j - \theta \). Clearly, \( t_j' \to +\infty \). Also \( z_j = S_{t_j'}x_j \in B_0 \) for large \( j \), and from this sequence we can extract a subsequence \( \{z_{j_k}\} \) converging to \( x \). Bearing (1.2) in mind we find that \( z \in \mathcal{A} \).

Now we verify that \( S_\theta \mathcal{A} = \mathcal{A} \). The reverse inclusion can be proved similarly.

We now come to the proof that \( \mathcal{A} \) is attracting. We show to begin with that

\[
1.3 \quad \text{dist}(S_tB_0, \mathcal{A}) \to 0 \quad \text{for} \quad t \to +\infty.
\]

Suppose that (1.3) is false. Then there are sequences \( \{x_j\} \subset B_0, \ t_j \to +\infty \), such that

\[
1.4 \quad \text{dist}(S_{t_j}x_j, \mathcal{A}) \geq \varepsilon > 0.
\]

From \( \{S_{t_j}x_j\} \) we can extract a subsequence that converges to a point \( x \). By (1.2), \( x \in \mathcal{A} \) and from (1.4) it follows that \( \text{dist}(x, \mathcal{A}) \geq \varepsilon \), which gives a contradiction. This proves (1.3). Since \( B_0 \) is an absorbing set, (1.3) implies that \( \mathcal{A} \) is attracting for any bounded \( B \).

that the \( S_t \) are uniformly bounded in \( t \in [0, T] \) for any fixed \( T \) it follows that the semigroup \( \{S_t\} \) is uniformly bounded on \( [0, +\infty) \).

**Theorem 1.2.** Suppose that a semigroup \( \{S_t\} \) is uniformly bounded in \( E \) and that there is a bounded absorbing set \( B_0 \). Suppose also that the operators \( S_t \) of \( \{S_t\} \) are completely continuous for \( t > 0 \). Then the semigroup \( \{S_t\} \) has a compact maximal attractor.

**Proof.** We set \( B_0 = S_tB_0 \). Since \( B_0 \) is bounded and \( S_t \) is completely continuous, \( B_0 \) is compact. Since \( B_0 \) is absorbing, so is \( B_0 \). For if \( B \) is bounded then \( S_tB \subset B_0 \) for \( t \geq T = T(B) \), consequently, \( S_tB = S_{t-T}B \subset S_tB_0 \) for \( t \geq T(B) + 1 \). Thus, all the conditions of Theorem 1.1 are satisfied, and it follows that \( \{S_t\} \) has a maximal attractor.

**Remark 1.1.** Let \( K \) be a closed subset of \( E \) that is weakly invariant under \( \{S_t\} \), that is, \( S_tK \subset K \) for \( t \geq 0 \). If the conditions of Theorem 1.1 are satisfied on \( K \), then the semigroup \( \{S_t \mid K\} \) has an attractor \( \mathcal{A}_K \). The proof remains the same.

If \( \{S_t \mid K\} \) satisfies the conditions of Theorem 1.2, in particular, if the \( S_t \) are completely continuous on \( K \), then \( \{S_t \mid K\} \) has an attractor \( \mathcal{A}_K \).
§ 2. Examples of parabolic equations and systems having a maximal attractor

In this section we construct semigroups for some classes of equations and use the results of § 1 to prove the existence of attractors for these semigroups. We consider parabolic equations with monotonic principal part, second-order parabolic equations, and systems of the type of chemical kinetics. Hyperbolic equations with dissipation are discussed in § 11. The two-dimensional Navier-Stokes system is investigated separately in § 8.

As usual, in what follows we denote by $W^p_2(\Omega)$ the Sobolev space (see [20]), by $H_0(\Omega)$ the space $W^p_2(\Omega)$, by $\| \|_p$ the norm in $H_0(\Omega)$ and by $\| \|$ that in $L_2(\Omega)$. We denote by $C^\alpha(\Omega)$ the space of functions on $\Omega$ satisfying a Hölder condition with the exponent $\alpha$ for $0 < \alpha < 1$ and a Hölder condition with the exponent $\alpha_1$ for $k$-th order derivatives if $\alpha = k + \alpha_1$, where $k$ is an integer and $0 < \alpha_1 < 1$. We denote by $V^\beta(0, T) = C^{\beta, \beta/2}_t$ the space of functions on the cylinder $\Omega \times [0, T]$ that satisfy (together with their derivatives) a Hölder condition and that are smooth of order $\beta$ in $x$ and of order $\beta/2$ in $t$; clearly, if $u(\cdot, \cdot) \in V^\beta(0, T)$, then $u(\cdot, t) \in C^\beta(\Omega)$ for all $t \in [0, T]$ and $u(x, \cdot) \in C^{\beta/2}(0, T)$ for all $x \in \Omega$.

For the properties of all these spaces and the corresponding embedding theorems, see [20]–[22].

**Example 2.1.** A parabolic equation on a torus with a monotonic principal part.

We consider the equation

$$
(2.1) \quad \partial_t u = \sum_{i=1}^n [\partial_i (a_i (\nabla u)) + b_i (x) \partial_i u] - f (x, u) + \lambda u \quad (\partial_i = \partial/\partial x_i),
$$

where $\nabla = \nabla_x$ and the coefficients are periodic with period $2\pi$ in each variable $x_i$. Hence we may assume that $x \in T^n$, where $T^n$ is an $n$-dimensional torus. We restrict ourselves to the case of a single second-order equation of the form (2.1), since the generalization to the corresponding parabolic systems and to higher-order equations is straightforward. We assume that the functions $a_i, b_i, f$ are twice continuously differentiable in all variables and that the principal part is strongly elliptic:

$$
(2.2) \quad \sum_{i,j=1}^n a_{ij} (\xi) \xi_i \xi_j \geqslant \mu_0 \sum_{i=1}^n \xi_i^2, \quad \mu_0 > 0 \quad (a_{ii} = \partial a_i/\partial x_i),
$$

$\forall \xi = (\xi_1, \ldots, \xi_n), \forall \xi^\prime = (\xi_1, \ldots, \xi_n)$, and also that it satisfies the condition

$$
(2.3) \quad \mu_1 (1 + |\xi|^2 + |\xi|^p) \geqslant \sum_{i=1}^n a_i (\xi) \xi_i \geqslant \mu_0 (|\xi|^p + |\xi|^2) \quad \forall \xi \in \mathbb{R}^n.
$$

On $f$ we also impose the conditions

$$
(2.4) \quad f_\prime (x, u) \geqslant 0 \quad (f_\prime = \partial f/\partial u),
$$

$$
(2.5) \quad \mu_1 |u|^p + C \geqslant f (x, u) \geqslant \mu_0 |u|^p - C, \quad p_0 > 2,
$$

$$
(2.6) \quad |\nabla_x f (x, u)| \leqslant C (1 + |u|)^{p_0/2}.
$$
Theorem 2.1. For any \( u_0 \in L_2(T^n) \) and any \( T > 0 \), the equation (2.1) with
the initial condition \( u(0) = u_0 \) has a unique solution that belongs to the
spaces
\[
L_\infty ((0, T); L_2(T^n)) \cap L_p, ((0, T); W^1_p, (T^n)) \cap L_{p*}((0, T); L_{p*}(T^n)),
\]
and for which
\[
(2.7) \quad \sup_{0 \leq t \leq T} || u(t) ||^2 + \frac{H_0}{2} \int_0^T \int_{T^n} (| \nabla u |^p + | \nabla u |^2 + | u |^p) \, dx \, dt \leq \|| u(0) ||^2 + C_0T,
\]
(2.8) \quad \| u(t) \|^2 \leq C_1 + (\| u(0) \|^2 - C_1) e^{-C't}, \quad C' > 0,
where \( C_0, C_1 \) and \( C' \) are independent of \( u_0 \) and \( T \), and
\[
(2.9) \quad \| \nabla u \|^2 + \frac{H_0}{2} \int_0^t \| \Delta u (\tau) \|^2 \, d\tau \leq C_2 \int_0^t (\| u \|^2 + \| u \|^p + \| u \|^p) \, d\tau,
\]
where \( C_2 \) is independent of \( u_0 \). The dependence of \( u(t) \) on \( t \) is weakly continuous in \( L_2(\Omega) \). The correspondence \( u_0 \to u(t) \) determines a one-
parameter semigroup \( \{ S_t \} \), \( S_t u_0 = u(t), S_t : L_2(T^n) \to L_2(T^n) \) and the
operators \( S_t \) are continuous.

Proof. The existence and uniqueness of the solution of the Cauchy problem
for (2.1) are well known and can be proved by standard methods of the
theory of monotonic operators (see, for example, [23] and [24]). The fact
that the linear terms violate monotonicity does not bring in essential
complications. We restrict our proof to a formal derivation of the a priori
estimates (2.7)-(2.9).

We obtain the estimate (2.7) by multiplying (2.1) by \( u \), integrating with
respect to \( x \) and \( t \) and using the inequalities (2.3) and (2.5). Here the
expression \( | \sum b_i \partial_i u \ | \) is bounded above by \( \frac{H_0}{2} | \nabla u |^2 + C_4 | u |^4 \), and
terms of the form \( (C_5 + \lambda) | u |^2 \) by \( \frac{H_0}{2} | u |^p + C_6 \). After elementary
transformations we arrive at (2.7).

To obtain (2.8) we multiply (2.1) by \( u(t) \), integrate with respect to \( x \), and
after similar transformations we obtain the differential inequality
\[
\partial_t \| u \|^2 + C' \| u \|^2 \leq C_1, \quad C' > 0.
\]
This yields (2.8). To derive (2.9) we multiply (2.1) by \( t^2 \Delta u \), integrate with
respect to \( x \) and \( t \), and integrate by parts. We find that
\[
(2.10) \quad \frac{1}{2} \partial_t \| t \nabla u \| - 2 (\nabla u, t \nabla u) + \sum_{i,j=1}^n t^2 (\partial_j a_i (\nabla u), \partial_i \partial_j u) +
\]
\[
+ 2 \sum_{j=1}^n (\partial_j f, \partial_j u) + \sum_{i=1}^n t^2 (b_i \partial_i u, \Delta u) + \lambda (u, \Delta u) t^2 = 0,
\]
where $\langle , \rangle$ is the scalar product in $L_2(T^n)$. Using the conditions (2.2), (2.4), (2.6), carrying out some simple calculations, and integrating (2.10) with respect to $t$ we obtain (2.9).

The weakly continuous dependence of $u(t)$ on $t$ in $L_2(T^n)$ follows from the fact that $u(t) \in L_\infty((0, T); L_2(T^n))$ and also that by (2.1) and the properties of $u(x, t)$, $\partial_t u$ belongs to $L_1((0, T); \left( W_q^s(T^n) \right)'$, where $q$ and $s$ are sufficiently large and $\left( W_q^s(T^n) \right)'$ is the dual space of $W_q^s$ (see [30] and [25]). The uniqueness of the solution of (2.1) and the continuous dependence of $u(t)$ on $u_0$ in $L_2(T^n)$ are well known and are proved in the standard way: the equations (2.1) for $u_1(t)$ and $u_2(t)$ with the initial conditions $u_{01}$ and $u_{02}$ are subtracted; the resulting equation for $u_1 - u_2$ is multiplied by $u_1 - u_2$ and integrated with respect to $x$ and $t$. After some simple calculations we arrive at the estimate $\|u_1(t) - u_2(t)\| \leq C \|u_{01} - u_{02}\|$, which implies the existence and continuity of $S_t$ (see, for example, [23] and [24]).

**Theorem 2.2.** The semigroup $\{S_t\}$ corresponding to (2.1) has the following properties: 1) $\{S_t\}$ is uniformly bounded in $L_2(T^n)$; 2) there is an absorbing set for $\{S_t\}$ that is bounded in $L_2(T^n)$; 3) the operators $S_t$ for $t > 0$ map bounded sets in $L_2(T^n)$ onto bounded sets in $H_1(T^n)$ and hence compact in $L_2(T^n)$.

**Proof.** If $B \subset L_2(T^n)$ lies within a ball $\{\|u\| \leq R\}$, then according to (2.8) $\|S_t u_0\|^2 = \|u(t)\|^2 \leq C_1 + R^2$ for all $t > 0$ and any $u_0 \in B$, that is, $\{S_t\}$ is uniformly bounded in $L_2$. Next, it follows from (2.8) that $\|S_t u_0\|^2 = \|u(t)\|^2 \leq 2C_1$, for $t \geq t_0 = t_0(R)$, that is, $\{u: \|u\|^2 \leq 2C_1\}$ is an absorbing set for $\{S_t\}$. This proves 1) and 2). To prove 3) we use (2.9). Since by (2.7) for $\|u_0\| \leq R$ the right-hand side of (2.9) is bounded by a constant depending only on $R$ and $T$, we find that $\|\nabla u\| \leq C_3(R, t)$ for $t > 0$. Bearing in mind that the embedding $H_1(T^n) \subset L_2(T^n)$ is compact, we obtain 3).

**Theorem 2.3.** The semigroup $\{S_t\}$, $S_t: L_2(T^n) \to L_2(T^n)$ corresponding to (2.1) has a maximal attractor $\mathcal{A}$, and $\mathcal{A}$ is bounded in $H_1(T^n)$.

**Proof.** The continuity of $S_t$ has been established in Theorem 2.1 and the other conditions of Theorem 1.2 follow from Theorem 2.2. Therefore, by Theorem 1.2 $\{S_t\}$ has a maximal attractor $\mathcal{A}$. This is bounded in $L_2(T^n)$, and since $S_t \mathcal{A} = \mathcal{A}$ for $t > 0$, by Theorem 2.2.3 $\mathcal{A}$ is bounded also in $H_1(T^n)$.

**Remark 2.1.** Theorems 2.1, 2.2, and 2.3 remain valid when the linear perturbations of the terms $\sum b_i \partial_i u + \lambda u$ that violate the monotonicity of (2.1) are replaced by a term $F_0(x, u, \nabla u)$ satisfying conditions that ensure that it is subordinate in (2.1). These conditions are dictated by the embedding theorems and restrict the growth of the derivatives of $F_0$. They are rather cumbersome and we do not quote them here.
Example 2.2. Second-order parabolic equation in a bounded domain.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial \Omega$. We consider the equation

$$
\partial_t u = \sum_{i,j=1}^{n} a_{ij}(x, u) \partial_i \partial_j u + b(x, u, \nabla u) - a_0(x, u) \equiv A(u), \quad u|_{\partial \Omega} = 0.
$$

(Instead of the boundary condition $u|_{\partial \Omega} = 0$ we can take other boundary conditions.) We assume that the ellipticity condition

$$
\sum a_{ij}(x, u) \xi_i \xi_j \geq \mu_0 |\xi|^2, \quad \mu_0 > 0
$$

is satisfied.

Suppose that the functions $a_{ij}$, $b$, and $a_0$ are three times continuously differentiable in all arguments and satisfy the conditions

$$
|b(x, u, \nabla u)| \leq C(u)(1 + |\nabla u|^2), \quad b(x, u, 0) = 0,
$$

and

$$
a_0(x, u) \operatorname{sign} u \geq \psi(|u|), \quad \psi(|u|) \to +\infty \quad \text{for} \quad |u| \to \infty,
$$

where $\psi(r)$ is a monotonic function. We look for a solution $u(x, t) = u(t)$ of (2.11) belonging to the space $V^{2+\alpha}(0, T)$, $0 < \alpha < 1$, for each $T > 0$ and satisfying the initial condition $u(0) = u_0$. By (2.11), compatibility conditions on $\partial \Omega$ arise at $t = 0$, which restrict $u_0$. To simplify these conditions we require that

$$
a_0(x, 0) = 0, \quad b(x, 0, \xi) = 0 \quad \text{for} \quad x \in \partial \Omega.
$$

The compatibility conditions on $u_0$ then take the form $u_0 \in E$, where

and the norm in $E$ is the same as in $C^{2+\alpha}$.

Theorem 2.4. Let $u_0 \in E$, where $E$ is defined by (2.16). Then (2.11) with the initial condition $u(0) = u_0$ has a unique solution. This solution belongs to $V^{2+\alpha}(0, T)$ for any $T > 0$ and is bounded in $V^{2+\alpha}(0, T)$ by a constant depending only on $\|u_0\|_E$ and on $T$. The correspondence $u_0 \to u(\cdot, t)$ determines a semigroup $\{S_t\}$ of maps $S_t : E \to E$, $t \geq 0$. The $S_t$ are continuous.

Proof. The existence, uniqueness and a priori estimates of solutions of (2.11) are proved in [22] (Ch. V, §6). Since $u(\cdot, \cdot) \in V^{2+\alpha}$, we see that $u(\cdot, t) \in C^{2+\alpha}$ and that $u(x, t)$ satisfies on $\partial \Omega$ the conditions in the definition of $E$, and by (2.15) also (2.11), that is, $u(\cdot, t) \in E$. The continuous dependence of $u(\cdot, t)$ on $u_0$ is proved in the usual way: the equations (2.11) for functions $u_1(t)$ and $u_2(t)$ corresponding to initial conditions $u_{01}$ and $u_{02}$ are subtracted. The resulting linear equation for $u_1 - u_2 = w$ has coefficients depending on $u_1$ and $u_2$ and belonging to $V^\alpha$. By standard theorems on the properties of parabolic equations (see [22])

$$
\|w(t)\|_E \leq C \|w(0)\|_E,
$$

and the continuity of $S_t$ follows.
Theorem 2.5. The semigroup \( \{S_t\} \) corresponding to (2.11) has the following properties: 1) \( \{S_t\} \) is uniformly bounded in \( E \); 2) there is an absorbing set for \( \{S_t\} \) that is bounded in \( E \); 3) the operators \( S_t \) for \( t > 0 \) map bounded sets \( B \subset E \) in \( C(\Omega) \) onto bounded sets in \( C^{3+\beta}(\Omega) \), \( 0 < \beta < 1 \), hence, are compact in \( E \).

Proof. We begin with the proof of 1). We prove first the uniform boundedness in \( V^2 + \alpha \), namely:

(2.17) \[ \|u(t)\|_c \leq \max (\|u(0)\|_c, \psi^{-1}(0)), \]

where \( u(t) \) is a solution of (2.11) in \( V^2 + \alpha \) with the initial condition \( u_0 \) and where \( \psi \) is the same function as in (2.14). We use the maximum principle. If \( (x_0, t_0) \) is a point where \( u(x, t) \) has a positive maximum on \( \Omega \times [0, T] \), then \( \nabla_x u(x_0, \ t_0) = 0 \) and \( \sum a_{ij} \partial_i \partial_j u(x_0, \ t_0) \leq 0 \). Therefore, by (2.11), (2.13), and (1.4)

(2.18) \[ \psi(u(x_0, \ t_0)) \leq a_0(u(x_0, \ t_0)) \leq -\partial_t u(x_0, \ t_0). \]

Since \( \partial_t u(x_0, \ t_0) \geq 0 \), it follows that \( u(x, t) \leq \max (\psi^{-1}(0), \|u_0\|_c) \) for \( 0 \leq t \leq T \). Using the same arguments for a point of a negative minimum, we obtain (2.17). We now observe that the solutions of (2.11) satisfy

(2.19) \[ \|u\|_{V^3 + 6(\tau_1 + \delta, \tau_1 + 2)} \leq K(\delta, \|u\|_{c(\alpha \times [\tau_1, \tau_1 + 2])}) \]

for any \( \delta, 0 < \delta < 1 \). (A proof of this estimate based on results of [22] is in [26].) Using (2.17) and (2.19) with \( \delta = 1/2 \) we find that \( u(t, x) \) is bounded in \( V^3 + \delta(\tau, \tau + 1) \) for any \( \tau > 1/2 \) by a constant independent of \( \tau \). Hence, using an estimate in \( V^{2+\alpha}(0, T) \) for a finite \( T \) we now obtain an estimate of \( u(t) \) in \( C^{2+\alpha}(\Omega) \) that is uniform in \( t \), and this completes the proof of 1).

To prove 2) we first demonstrate the existence of an absorbing set for \( \{S_t\} \) that is bounded in \( C \). We claim that

(2.20) \[ \forall \ x \in \Omega \ |u(x, 0)| \leq R, \ \text{then} \ \forall \ t \geq 2R, \ |u(x, t)| \leq \psi^{-1}(1) \]

for any solution of (2.11) in \( V^2 + \alpha \). We sketch only the main features of the proof of (2.20). Let \( (x_0, t) \) be a point of a positive maximum of \( u(x, t) \) for fixed \( t \). We denote the value of this maximum by \( \varphi(t) \). Clearly, (2.18) holds at \( (x_0, t) \), where \( t_0 = t \). If \( \partial_t u(x_0, t) \geq -1 \) at any such \( x_0 \), then \( \varphi(t) < \psi^{-1}(1) \). If the latter inequality holds for some \( t = t_1 \), where \( t_1 \in [0, 2R] \), then we find by analogy with (2.17) (where \( u(0) \) is replaced by \( u(t_1) \)) that \( u(x, t) \leq \psi^{-1}(1) \) for \( t \geq t_1 \). It remains to prove that such a \( t_1 \) always exists. Suppose that this is false, that is, \( \partial_t u(x_0, t) < -1 \) for any \( t \in [0, 2R] \) and any maximum point \( x_0 \) for this \( t \). Simple arguments show that then \( \varphi(t + \tau) \leq \varphi(t) - 2\tau/3 \) for \( 0 \leq \tau \leq \epsilon \), where \( \epsilon = \epsilon(t) \) is sufficiently small. From this it is not hard to deduce, since \( \varphi(0) \leq R \), that \( \varphi(t) \leq R - 2\tau/3 \) and \( \varphi(2R) \leq -R/3 < 0 \), which contradicts the assumption that \( \varphi \) is positive. Considering the negative minima along the same lines, we obtain (2.20),
from which it follows that there is an absorbing set that is bounded in $C(\Omega)$. Using (2.19) we then find that there is an absorbing set bounded in $C^{3+\beta}(\Omega)$, which implies 2).

Now 3) follows directly from (2.19) and the compactness of the embedding $C^{3+\beta}(\Omega) \subset C^{2+\alpha}(\Omega)$.

**Theorem 2.6.** The semigroup $\{S_t\}$, $S_t : E \to E$, where $E$ is defined by (2.16), has a maximal attractor $\mathcal{A}$, which is bounded in $C^{3+\beta}$, $0 < \beta < 1$.

**Proof.** The existence of an attractor follows from Theorems 2.5 and 1.2; the boundedness of $\mathcal{A}$ in $C^{3+\beta}$ from its boundedness in $C^{2+\alpha}$ and from Theorem 2.5.3), since $\mathcal{A} = S_t \mathcal{A}$ $\forall \ t > 0$.

**Example 2.3.** The parabolic system of the type of chemical kinetics and its generalizations.

In a bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary $\partial \Omega$ we consider the system

\[\frac{\partial}{\partial t} u = a \Delta u + \sum_{i=1}^{n} \partial_i (b_i(x,u)) - f(x,u) + \lambda u, \quad u |_{\partial \Omega} = 0,\]

where $u = (u^1, \ldots, u^m)$, $f = (f^1, \ldots, f^m)$, $b_i = (b_i^1, \ldots, b_i^m)$, and $a$ is a matrix of order $m$ with constant coefficients whose symmetric part $a_+ = (a^*_+ + a_-)/2$ is positive definite: $a_+ \geq \mu_0 I$, $\mu_0 > 0$, and $\Delta$ is the Laplace operator (see, for example, [36], [33], and [15]). The functions $b_i$ and $f$ are assumed to be twice continuously differentiable in all variables and to satisfy the conditions

\[\sum_{i, \lambda=1}^{m} \frac{\partial f_{\lambda}}{\partial u_i} b_{k\lambda} e_i \geq \epsilon.\]

(Since (2.21) contains a large parameter $\lambda$, it is sufficient that (2.24) holds for $\lambda = 1$, where $\lambda > 0$.)

**Theorem 2.7.** For any $u_0 \in (L_2(\Omega))^m$ the system (2.21) with the initial condition $u(0) = u_0$ has a unique solution $u(t)$ belonging to $L_\infty\times L_2(\Omega)$ $\cap$ $L_2((0, T); (L_2(\Omega))^m)$ $\cap$ $L_{p_1}(\Omega)^m$ $\cap$ $L_{p_2}((0, T); (L_{p_1}(\Omega))^m)$.

Here (2.7), (2.8), and (2.9) hold, where $p_1 = 2$; $u(t)$ depends weakly continuously on $t$ in $(L_2(\Omega))^m$. The correspondence $u_0 \to u(t)$ determines a semigroup $\{S_t\}$, $S_t : (L_2(\Omega))^m \to (L_2(\Omega))^m$, and the operators $S_t$ are continuous.

The proof is similar to that of Theorem 2.1 and uses the monotonicity and subordination of the first-order terms.
Theorem 2.8. The semigroup \( \{S_t\} \) corresponding to (2.21) has the following properties: 1) \( \{S_t\} \) is uniformly bounded in \((L_2(\Omega))^m\); 2) \( \{S_t\} \) has an absorbing set that is bounded in \((L_2(\Omega))^m\); 3) the operators \( S_t \) for \( t > 0 \) map bounded sets in \((L_2(\Omega))^m\) to bounded sets in \((H_1(\Omega))^m\) and are therefore compact in \((L_2(\Omega))^m\).

The proof is quite analogous to that of Theorem 2.2.

Theorem 2.9. The semigroup \( \{S_t\}, S_t: (L_2(\Omega))^m \to (L_2(\Omega))^m \), corresponding to (2.21) has a maximal attractor \( \mathcal{A} \) that is bounded in \((H_1(\Omega))^m\).

The proof is the same as for Theorem 2.3.

Remark 2.2. Theorems 2.7, 2.8, and 2.9 remain valid if \( a\Delta \) in (2.1) is replaced by an elliptic second-order differential operator \( \mathcal{L} \) satisfying certain conditions. The condition of monotonicity on \( f(u) \) can also be weakened.

Remark 2.3. For real systems of chemical kinetics, \( \{S_t\} \) acts invariantly on the cone \( K = \{ u = (u^1, \ldots, u^m): u^j(x) \geq 0, j = 1, \ldots, m; x \in \Omega \} \) (see [37]). If the conditions (2.22)-(2.24) are satisfied, then the semigroup \( \{S_t\} \) restricted to \( K \) has an attractor \( \mathcal{A}_K \) (see Remark 1.1). Obviously, \( \mathcal{A}_K \subset \mathcal{A} \cap K \), where \( \mathcal{A} \) is the maximal attractor constructed in Theorem 2.9.

§3. The Hausdorff dimension of invariant sets

In this section we introduce the concept of a \((k, N)\)-contracting map (see [42] and [47]). We show that if a compact set \( X \) in \( H \) is invariant under a \((k, N)\)-contracting map for small \( k \), then the Hausdorff dimension of \( X \) does not exceed \( N \). A similar property holds for flattening maps.

Let \( H \) be a Hilbert space with norm \( \| \cdot \| \). We denote by \( B_r(x) \) the closed ball of radius \( r \) in \( H \) with the centre at \( x \). Let \( H^N \) be an \( N \)-dimensional subspace of \( H \) and \( \pi^N \) be the orthogonal projection operator onto \( H^N \). We denote by \( E^N \) the ellipsoid in \( H^N \) with centre at the origin and semi-axes \( \alpha_1, \ldots, \alpha_N, \alpha_j > 0 \), labelled in decreasing order. Let \( \Lambda^N = \Lambda^N(\alpha_1, \ldots, \alpha_N) \) be the set:

\[
\Lambda^N = \{ u \in H: u = u_1 + u_2, u_1 \in E^N, u_2 \in (I - \pi^N)H, \| u_2 \| \leq \alpha_N \},
\]

We call such sets bicylinders. Clearly, \( \Lambda^N \) is the product of the ellipsoid \( E^N \) in \( H^N \) with a ball of radius \( \alpha_N \) in the orthogonal complement to \( H^N \), and the radius of this ball is the minimal axis of the ellipsoid. We denote by \( \omega_N(\Lambda^N) \) the number

\[
\omega_N(\Lambda^N(\alpha_1, \ldots, \alpha_N)) = \alpha_1 \ldots \alpha_N,
\]

which up to a constant factor is equal to the volume of \( E^N \).

Definition 3.1. Let \( X \subset H, N \geq 0, N \in \mathbb{N}, \) and \( k > 0 \). A map \( S: X \to H \) is said to \((k, N)\)-contract the set \( X \) in \( H \) if there exists an \( R_0 > 0 \) such that for
any \( x \in X \) there is a projection operator \( \pi^N = \pi^N(x) \) onto an \( N \)-dimensional subspace \( H^N = H^N(x) \subseteq H \) such that for \( r < R_0 \)
\[
S(X \cap B_r(x)) \subseteq S(x) + r\Lambda^N(\alpha_1, \ldots, \alpha_N),
\]
and
\[
\omega_N(\Lambda^N(\alpha_1, \ldots, \alpha_N)) \leq k.
\]

**Remark 3.1.** If the operator \( S \) is linear, \( X = H \), and \( S \) is a \((k, N)\)-contracting map of \( H \) into \( H \), then the projection operator \( \pi^N \) can clearly be chosen to be independent of \( x \), that is \( \pi^N(x) = \pi^N(0) \) and \( R_0 \) is arbitrary.

Let us recall the definition of the Hausdorff dimension of a set \( X \subseteq H \). Let \( U \) be a covering of \( X \) by balls \( B_r(x_j) \) of radii \( r_j \leq \delta \). We set
\[
h_N(U, \delta, X) = \sum_{j} r_j^N, \quad h_N(\delta, X) = \inf h_N(U, \delta, X),
\]
where the lower bound is taken over all coverings \( U \) by balls of radii not exceeding \( \delta \). We define the *Hausdorff measure of dimension \( N \) of \( X \) by*
\[
\lim_{\delta \rightarrow 0} h_N(\delta, X) = h_N(X).
\]
The Hausdorff dimension \( \dim X \) of \( X \) is defined by
\[
\dim X = \inf \{ N : h_N(X) < \infty \}.
\]

**Theorem 3.1.** Let \( X \) be a compact set in \( H \) and \( S : X \rightarrow X \) a map such that \( S(X) = X \). Then for any integer \( N > 0 \) there is a constant \( C^0_N \) depending only on \( N \) and such that if \( S \) \((k, N)\)-contracts \( X \) in \( H \) and \( k < (C^0_N)^{-1} \), then the Hausdorff dimension of \( X \) in \( H \) does not exceed \( N \):
\[
\dim X \leq N.
\]

**Lemma 3.1** (see [42]). Let \( \Lambda^N = \Lambda^N(\alpha_1, \ldots, \alpha_N) \) be the bicylinder defined by (3.1). Then there is a covering of \( \Lambda^N \) by balls of radius \( \rho \)
\[
\rho \leq \lambda_N(\omega_N(\Lambda^N))^{1/N}, \quad \lambda_N = (N + 1)^{1/2},
\]
and the number \( n \) of these balls is subject to
\[
n \leq C_N \rho^{-N} \omega_N(\Lambda^N), \quad C_N = 2^N(N + 1)^{N/2}.
\]

**Proof.** The projection of \( \Lambda^N \) onto \( H^N \) is the ellipsoid \( E^N \). We inscribe this ellipsoid in a parallelepiped \( P \) with sides of lengths \( 2\alpha_i \) \((i = 1, \ldots, N)\). We can now cover \( P \) with \( n \) cubes \( K_j \) of side \( 2\rho_0 \), where \( \rho_0 = \alpha_N \) and
\[
n = \prod_{i=1}^{N} [(\alpha_i/r_0) + 1]. \quad \text{Since} \ \alpha_i/\rho_0 \geq 1,
\]
\[
n \leq 2^N \prod_{i=1}^{N} (\alpha_i/\rho_0) = 2^N \rho_0^{-N} \omega_N(\Lambda^N).
\]

Clearly, the sets \( M_j = K_j \times (I - \pi^N)B_{\rho_0}(0) \) form a covering of the bicylinder \( \Lambda^N \subseteq H \). Such sets \( M_j \) are contained within a ball of radius \( \rho = \rho_0(N+1)^{1/2} = \lambda_N \rho_0 \). Since \( \rho_0 = \alpha_N \leq (\alpha_1, \ldots, \alpha_N)^{1/N} = (\omega_N(\Lambda^N))^{1/N} \), (3.5) holds. Substituting \( \rho_0 = \rho/\lambda_N \) in the right-hand side of (3.7) we obtain (3.6).
Lemma 3.2. Let $X$ be compact in $H$ and $S : X \to H$ a $(k, N)$-contracting map. Then for $e < R_0/2$

\begin{equation}
    h_N(2\lambda_N k^{1/N}e, \ S(X)) \leq k2^N C_N h_N(e, X).
\end{equation}

Proof. Let $U$ be a covering of $X$ by balls $B_{r_j}(y_j), r_j < e < R_0/2$ ($R_0$ is the number in Definition 3.1). It is easy to see that by taking $z_j \in B_{r_j}(y_j) \cap X_f$ and balls $B_{2r_j}(x_j)$ we obtain a new covering of $X$ by balls with centres at $x_j \in X$. By (3.3), the image of each $B_{2r_j}(x_j) \cap X$ under $S$ is contained within the bicylinder $S(x_j) + 2r_j \Lambda^N$, where $\omega_N(\Lambda^N) \leq k$. Using Lemma 3.1 we can now cover each bicylinder $\Lambda^N$ with $n_j$ balls of radius $\rho_j$. By (3.5) and (3.4), we can take $\rho_j \leq \lambda_N k^{1/N}$. The $n_j$ balls of radius $r_j = 2\rho_j e_j$ form a covering of $S(x_j) + 2r_j \Lambda^N$. Clearly,

\begin{equation}
    r_j \leq \lambda_N k^{1/N} 2e, \quad r_j \leq e.
\end{equation}

Since the $S(x_j) + 2r_j \Lambda^N$ form a covering of $S(X)$, we can now cover these bicylinders with balls as outlined above, which gives a covering of $S(X)$ denoted by $U_f$. Using (3.6) to estimate $n_j$ we find that

\begin{equation}
    h_N(U_f, \lambda_N k^{1/N} 2e, S(X)) = \sum_j n_j (r_j)^N \leq C_N \sum_j \rho_j^{-N} \omega_N(\Lambda^N) (r_j)^N \leq kC_N \sum_j \rho_j^{-N} (r_j)^N = kC_N 2^N \sum_j r_j^N = k2^N C_N h_N(U, e, X).
\end{equation}

This yields (3.8).

Proof of Theorem 3.1. Since $S(X) = X$, from (3.8) we find that

\begin{equation}
    h_N(2\lambda_N k^{1/N}e, X) \leq k2^N C_N h_N(e, X).
\end{equation}

If $k < (2\lambda_N)^{N/2}$ and $k < (C_N 2^N)^{-1/2}$, then it follows that $h_N(e/2, X) \leq (1/2)h_N(e, X)$. As $e \to 0$, we obtain $h_N(X) = 0$, that is, $\dim X < N$.

Remark 3.2. The assertion of Theorem 3.1 can be generalized to non-integral $N$ (for more detail, see [42]). The corollaries obtained in §5 also generalize to non-integral $N$. For simplicity, we restrict ourselves in what follows to integral $N$.

We now explain a slightly different approach to an estimate of the Hausdorff dimension (see [46]).

Definition 3.2. Let $X \subset H, M > 0, \text{ and } e(N) \to 0 \text{ as } N \to \infty$. A map $S : X \to H$ is said to flatten $X$ in $H$ with the flattening constants $(M, e(N))$ if there exists an $R > 0$ such that for any $x \in X$ there is an orthogonal projection $\pi^N(x)$ onto an $N$-dimensional subspace $H^N = H^N(x) \subset H$ such that

\begin{equation}
    S(X \cap B_r(x)) \subset S(x) + \pi^N(B_M(0)) + (I - \pi^N)B_{e(N)}(0)
\end{equation}

for $r < R(N)$.

Theorem 3.2 (see [13]). There is a constant $C^0$ such that if $X$ is compact, $S(X) = X$, and $S$ flattens $X$ in $H$, then from $Me(N) \leq C_0$ it follows that

\begin{equation}
    \dim X < 2N.
\end{equation}
We mention that the flattening condition in a number of cases is easier to verify than the condition for \((k, N)\)-contraction.

We now derive a simple sufficient condition for a map to be flattening.

**Lemma 3.3.** Suppose that \(H_1\) is a separable Hilbert space with norm \(\| \cdot \|_1\), that \(H_1 \subset H\), and that this embedding is completely continuous and \(H_1\) is everywhere dense in \(H\). Let \(X \subset H\) and suppose also that \(S\) maps \(X\) into \(H_1\) and that the Lipschitz condition

\[
|S(x_1) - S(x_2)|_1 \leq M_1 \| x_1 - x_2 \| \quad \forall x_1, x_2 \in X,
\]

holds, where \(M_1\) is independent of \(x_1\) and \(x_2\). Then \(S\) flattens \(X\) in \(H\).

**Proof.** Let \(\{e_i\}\) be an orthogonal basis in both \(H\) and \(H_1\) that is orthonormal in \(H_1\): \(\| e_i \|_1 = 1\). We assume that the \(e_i\) are labelled so that \(\| e_i \|\) decreases for increasing \(i\). We can then choose \(H^N\) to be the subspace with the basis \(\{e_1, \ldots, e_N\}\). Since \(\| e_i \| \to 0\) as \(i \to \infty\), (3.10) follows from (3.11), where \(e(N) = M_1 \| e_{N+1} \|\).

We note that Lemma 3.3 also holds when \(H_1\) is a reflexive Banach space with a basis rather than a Hilbert space. In this case the results of [27] are used in the proof.

§4. Estimate of the change in volume under the action of shift operators generated by linear evolution equations

In this section we consider a certain class of linear evolution equations subject to natural conditions. The shift operator \(G_t\), which maps the initial

We obtain quantitative estimates of the change in the volumes under the action of \(G_t\). Subsequently these estimates are used to estimate the constant \(k\) of non-linear \((k, N)\)-contracting shift operators \(S_t\) and to estimate the dimension of attractors.

Let \(H\) be a Hilbert space, \(Q\) a linear self-adjoint positive definite operator whose domain \(D(Q) \subset H\) is dense in \(H\). We require that \(Q^{-1}\) is completely continuous. We denote by \(H_s\) the scale of Hilbert spaces generated by \(Q\). The scalar product in \(H_s\) is defined by \((u, v)_s = (Q^s u, v)\). It is obvious that \(H_0 = H\), that \(H_s \subset H_\sigma\) for \(\sigma \leq s\), and that this embedding is completely continuous for \(s > \sigma\). Let \(\{e_i, i = 1, 2, \ldots\}\) be an orthonormal basis in \(H\) consisting of the eigenvectors of \(Q\) labelled in increasing order of the eigenvalues \(\gamma_i\):

\[
Qe_i = \gamma_i e_i, \quad 0 < \gamma_1 \leq \gamma_2 \leq \ldots
\]

Obviously, \(\{e_i\}\) is an orthogonal basis in each \(H_s\). We denote by \(E_N\) the \(N\)-dimensional subspace with the basis \(\{e_1, \ldots, e_N\}\) and by \(\Pi_N\) the orthogonal projection of \(H\) onto this subspace. Obviously, \(\Pi_N\) is an orthogonal projection in each \(H_s\) and its norm in each \(H_s\) is 1.
In this section we consider the linear equation

\[ \partial_t v(t) = L(t)v(t), \quad v(0) = v_0, \]

where \( L(t) \) is a linear operator in \( H_s \) depending on a parameter \( t \). We are interested in the properties of the shift operator \( G_t \) which for a given \( t \) assigns to each initial value \( v_0 \) the value \( v(t) \) of the corresponding solution of (4.2): \( G_tv_0 = v(t) \). In what follows we consider equations (4.2), where \( L(t) \) satisfies the following condition.

**Condition 4.1.** For every \( t \in [0, T] \) the operator \( L(t) \) maps \( H_x \) into \( H_y \), and

\[ ||L(t)v||_1 \leq C_1 ||v||_1 \quad \forall v \in H_1, \]

where \( C_1 \) is independent of \( t \in [0, T] \) and of \( v \in H_1 \);

\[ (L(t)v, v) \leq -C_2 ||v||_1^2 + C_3 ||v||_2^2 \quad \forall v \in H_4, \]

where \( C_2, C_3 > 0 \) are independent of \( t \in [0, T] \) and of \( v \in H_4 \);

(4.5) the functions \( (L(t)e_i, e_j) \) are continuous for \( t \in [0, T] \) and any eigenvectors \( e_i \) and \( e_j \) of \( Q \).

We use the representation of the solution of (4.2) as the limit of Galerkin approximations \( v^N(t) \), which are defined as the solution of the system

\[ \begin{align*}
\partial_t v^N(t) &= \Pi_N L(t)v^N(t), \\
v^N(0) &= \Pi_N v_0, \quad N = 1, 2, \ldots,
\end{align*} \]

where \( v^N \in E_N \) for each \( t \geq 0 \). It is obvious that (4.6) is a finite system of linear equations with coefficients \( (L(t)e_i, e_j) \).

**Theorem 4.1.** Suppose that an operator \( L(t) \) satisfies Condition 4.1. Then for any \( v_0 \in H \) there is a unique solution of (4.2) belonging to \( L^2((0, T); H_1) \cap C([0, T]; H_0) \cap H_4((0, T); H_{-1}) \). This defines an operator \( G_t : H \to H, \ v_0 \to v(t) = G_tv_0, \ t \in [0, T] \) such that

\[ ||v(t)|| \leq C_0 ||v_0|| \]

for \( 0 \leq t \leq T \), where \( C_0 \) is independent of \( t \in [0, T] \) and of \( v_0 \). If finitely many initial conditions \( v_{0i} \in H \ (i = 1, \ldots, k_0) \) are given, then there is a sequence of indices \( N_i \) such that the Galerkin approximations \( v^N_i(t) \) converge to \( v_i(t) \) in \( H \) for all \( i = 1, \ldots, k_0 \) almost everywhere in \( [0, T] \) as \( j \to \infty \).

The proof proceeds by a standard method (see, for example, [24] and [30]). The convergence almost everywhere in \( H \) follows from the convergence of \( v^N_i \) in \( L^2((0, T); H) \).

As was shown in [42] and [47], study of the change of finite-dimensional volumes under \( G_t \) is very important in estimates of the Hausdorff dimension of attractors.
We introduce the following notation. If $A : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator, then we set

\begin{equation}
\hat{\mu}_n(A) = \sup_{V_n \subset \mathcal{H}} \frac{\mu_n(A(B))}{\mu_n(B)}, \quad B \subset V_n,
\end{equation}

where $V_n$ is an arbitrary linear $n$-dimensional subspace of $\mathcal{H}$ and $\mu_n$ is the $n$-dimensional measure induced by the scalar product in $\mathcal{H}$ on the $n$-dimensional linear subspace $V_n$. Here $B$ is a measurable subset of $V_n$ of non-zero measure, for example, a unit ball in $V_n$ (clearly, $\mu_n(A(B))/\mu_n(B)$ is independent of the choice of $B$). It is evident that

\begin{equation}
\hat{\mu}_n(A \Lambda) \leq \hat{\mu}_n(A_1 \Lambda) \hat{\mu}_n(A_2).
\end{equation}

**Proposition 4.1.** Let $A : \mathcal{H} \to \mathcal{H}$ be a linear completely continuous operator and $B_1 = \{u \in \mathcal{H} : \|u\| \leq 1\}$. Then the set $A(B_1)$ is an ellipsoid and can be represented in the form

$$A(B_1) = \{u \in \mathcal{H} : u = \sum_i \xi_i g_i, \sum_i \alpha_i^{\xi_i} \leq 1\},$$

where the $g_i$ are orthonormal eigenvectors of $(AA^*)^{1/2}$ corresponding to eigenvalues $\alpha_i > 0$ (that is, the $\alpha_i$ are $s$-numbers of the operator $A$; see [31]).

**Proof.** For brevity we restrict ourselves to operators $A$ with zero kernel. Then $A(B_1)$ is defined by the condition $(A^{-1}u, A^{-1}u) \leq 1$, which can be re-written as $((AA^*)^{-1}u, u) \leq 1$. Expanding $u$ in the eigenbasis $\{g_i\}$ of $AA^*$ we obtain the required representation for $A(B_1)$.

Then $\hat{\mu}(A) = \alpha_1 \ldots \alpha_n$, where $\alpha_1, \ldots, \alpha_n$ are the eigenvalues of $(AA^*)^{1/2}$.

**Proof.** If $V_n$ is a linear $n$-dimensional subspace of $\mathcal{H}$, then $\mu_n(B_1 \cap V_n)$ is independent of $V_n$ and $\mu_n(A(B_1 \cap V_n)) = \mu_n(A(B_1) \cap A(V_n))$ reaches its maximum when the linear subspace $A(V_n)$ contains the $n$ semi-axes of the ellipsoid $A(B_1)$ of maximal length. It is obvious that then $\mu_n(A(B_1) \cap A(V_n))/\mu_n(B_1) = \alpha_1 \ldots \alpha_n$, which completes the proof.

**Proposition 4.3.** Let $A : \mathcal{H} \to \mathcal{H}$ be a linear completely continuous operator. Then $A$ $(k, n)$-contracts $\mathcal{H}$ in $\mathcal{H}$ if $k > \hat{\mu}_n(A)$, $k > 0$.

**Proof.** For $\pi^a$ in Definition 3.1 we take the orthogonal projection with the basis of vectors parallel to the $n$ maximal semi-axes of the ellipsoid $A(B_1)$ (see Definition 4.1). Here it is assumed that $\alpha_n \neq 0$. In that case the assertion follows at once from Proposition 4.2 and Definition 3.1. If $\alpha_n = 0$, then $\hat{\mu}_n(A) = 0$. Taking any $k > 0$ and replacing the zero axes of the ellipsoid with indices not exceeding $n$ by axes of sufficiently small length, we arrive at the required conclusion also in this case.
Theorem 4.2. Suppose that an operator $L(t)$ satisfies Condition 4.1 and that

\[(4.11) \quad (L(t)v, v) \leq ((-v(t)Q + h(t)I)v, v) \quad \forall v \in H_1,
\]

where $h(t)$ and $v(t)$ are integrable positive functions and $v(t)$ is continuous.

Let $G_{11} : H \to H$ be the shift operator generated by (4.2), where $L = L_1$. Then for each $n = 1, 2, \ldots$ and all $t \in [0, T]$.

\[(4.12) \quad \hat{\mu}_n (G_{11}) \leq \exp \left[ -\nu_1 T \text{Tr}_n (Q) + nh_1 \right],
\]

where

\[(4.13) \quad \nu_1 = \frac{1}{t} \int_0^t v(\tau) \, d\tau, \quad h_1 = \frac{1}{t} \int_0^t h(\tau) \, d\tau, \quad \text{Tr}_n (Q) = \sum_{i=1}^n \gamma_i,
\]

and the $\gamma_i$ are the eigenvalues of $Q$ (see (4.1)).

Proof. We write $L^N_{11}(t) = \Pi_N L(t)$ and consider the Galerkin system (4.6) corresponding to $L = L_1$. By (4.5), $L^N_{11}(t)$ depends continuously on $t$. From (4.11), taking $v \in \Pi_N H$, we obtain

\[(4.14) \quad (L^N_{11}(t)v, v) + v(t)(Qv, v) \leq h(t) (v, v) \quad \forall v \in \Pi_N H, \quad ||v|| = 1.
\]

We set $h_N(t) = \sup \{(L^N_{11}(t)v, v) + v(t)(Qv, v) : v \in \Pi_N H, \quad ||v|| = 1\}$. Clearly, the function $h_N(t)$ is continuous, $h_N(t) \leq h(t)$, and

\[(L^N_{11}(t)v, v) \leq ((-v(t)Q + h_N(t)I)v, v).
\]

Writing $L^N_{11}(t) = \Pi_N((-v(t)Q + h_N(t)I)v)$ we have

\[(4.15) \quad (L^N_{11}(t)v, v) \leq (L^N_{12}(t)v, v) \quad \forall v \in \Pi_N H.
\]

Next we conduct the proof of the theorem in two steps. In the first step we prove (4.12) for the operator $G^N_{11}$ corresponding to the Galerkin system (4.6). We consider first the case when the operators $L^N_{11}(t)$ are independent of $t$: $L^N_{11}(t) = L^N_{11}(t_0)$ $(i = 1, 2)$ and symmetric: $L^N_{11}(t_0) = (L^N_{11}(t_0))^*$. We than have $G^N_{11} = \exp(tL^N_{11}(t_0))$. By Proposition 4.2,

\[(4.16) \quad \hat{\mu}_n (e^{tL^N_{11}(t_0)}) = \prod_{j=1}^n e^{t \beta_j^1},
\]

where the $\beta_j^1$ are the eigenvalues of $L^N_{11}(t_0)$, labelled in decreasing order. We now note that according to Courant's minimax principle (see, for example [31]),

\[(4.17) \quad \beta_j^1 = \sup_{W_j} \inf_{u \in W_j, \quad ||u|| = 1} (L(t_0)u, u),
\]

where $W_j$ is a $j$-dimensional subspace. It follows from (4.15) and (4.11) that $\beta_j^1 \leq \beta_j^2$, and by (4.14) we obtain

\[(4.18) \quad \hat{\mu}_n (e^{tL^N_{11}(t_0)}) \leq \hat{\mu}_n (e^{tL^N_{12}(t_0)}) \quad \forall t_0 \in [0, T].
\]

We now consider an operator $L^N_{11}(t)$ that is not self-adjoint, is finite-dimensional, and depends on $t$. We denote by $L^N_{11}(t)$ the symmetric and by $L^N_{11}(t)$ the skew-symmetric part of $L^N_{11}(t)$: $L^N_{11}(t) = L^N_{11}(t) + L^N_{11}(t)$.
By the Trotter-Daletskii formula applied to the finite-dimensional case,

\[(4.17) \quad G_{1t}^N = \lim_{m \to \infty} \prod_{j=1}^{m} e^{\Delta t L(t_j)} e^{\Delta t L^{-1}(t_j)},\]

where \(\Delta t = t/m\) and \(t_j = jt/m\). It is obvious that

\[(4.18) \quad \hat{\mu}_n (G_{1t}^N) = \lim_{m \to \infty} \hat{\mu}_n \left( \prod_{j=1}^{m} e^{\Delta t L(t_j)} e^{\Delta t L^{-1}(t_j)} \right).\]

The operators \(U_j = \exp(\Delta t L(t_j))\) are isometric. Therefore, from (4.10) and (4.16) it follows that

\[(4.19) \quad \hat{\mu}_n (G_{1t}^N) \leq \prod_{j=1}^{m} \hat{\mu}_n (e^{\Delta t L(t_j)}) \leq \prod_{j=1}^{m} \hat{\mu}_n (e^{\Delta t L^2(t_j)}).\]

We note that the eigenvalues \(\beta_i^2\) of \(L_N(t_j) = -v(t_j)\Pi_N Q + h_N(t_j)I\) are given by

\[\beta_i^2 = h_N(t_j) - \gamma_i v(t_j).\]

Using (4.14) with \(t = At\) and \(i = 2\) we obtain

\[(4.20) \quad \prod_{j=1}^{m} \hat{\mu}_n (e^{\Delta t L_N(t_j)}) = \exp \left( -\sum_{j=1}^{m} v(t_j) \Delta t \operatorname{Tr}_n (Q) + \sum_{j=1}^{m} h_N(t_j) \Delta t n \right).\]

Taking the limit in (4.18), using (4.19) and (4.20) with the notation (4.13), and bearing in mind that \(h_N(t) \leq h(t)\), we conclude that

\[(4.21) \quad \hat{\mu}_n (G_{1t}^N) \leq \exp \left( -v_1 \operatorname{Tr}_n (Q) + nh_1 \right).\]

Let us now prove (4.12) for an infinite-dimensional operator \(G_{1t}\). For \(B\) in the definition of \(\hat{\mu}_n\) we take the \(n\)-dimensional parallelepiped constructed from the vectors \(v_1, ..., v_n\). Under the action of \(G_{1t}\) these vectors go over into \(v_1(t), ..., v_n(t)\). Clearly,

\[v_n(t) \to v(t) \text{ in } H \text{ for almost all } t \text{ as } N \to \infty. \]

By Theorem 4.1, there is a subsequence of Galerkin approximations \(v_N(t) \to v(t)\) in \(H\) for almost all \(t\) as \(N \to \infty\). By (4.21),

\[(4.23) \quad |\det \{(v_N^N(t), v_N^n(t))\}|^{1/2} |\det \{(v_N^N(0), v_N^n(0))\}|^{1/2} \leq \exp \left( -v_1 \operatorname{Tr}_n (Q) + nh_1 \right).\]

Since \(v_N(t) \to v(t)\) in the norm in \(H\), the left-hand side of (4.23) tends to the right-hand side of (4.22). Therefore, for almost all \(t\)

\[(4.24) \quad \mu_n (G_{1t} (B)) \leq \exp \left( -v_1 \operatorname{Tr}_n (Q) + nh_1 \right).\]

Since \(v(t) \in C([0, T], H)\), it follows from (4.22) that the left-hand side of (4.24) depends continuously on \(t\): this yields at once (4.12) for all \(t\) and completes the proof.

The following simple assertion holds:

**Proposition 4.4.** Let \(A : H \to H\) be a bounded linear operator such that \((\hat{\mu}_n(A))^{1/n} \to 0\) as \(n \to \infty\). Then \(A\) is completely continuous.

**Proof.** We consider the self-adjoint operator \(AA^*\). We denote by \(H_0\) the subspace \((I - E_0)H = H_0\), where \(E_0\) is the spectral projection corresponding...
to the interval $(0, \delta)$ of the spectrum and generated by the operator $AA^*$; $\delta > 0$. It is obvious that $H_\delta$ is invariant under $AA^*$ and that $(AA^*v, v) \geq \delta \|v\|^2$ for all $v \in H_\delta$. Therefore, the operator $AA^*$ is invertible on $H_\delta$. We now claim that from $(\mu_\eta(A))^{1/n} \to 0$ as $n \to \infty$ it follows that $H_\delta$ is finite-dimensional. For suppose that there is an $n$-dimensional subspace $\mathcal{Y} \subset H_\delta$. The intersection of this subspace with $B_0 = \{u \in H_\delta^0: ((AA^*)^{-1}u, u) \leq 1\}$ is an ellipsoid, which can be described as $B_0 = \{u \in H_\delta: ((AA^*)^{-1}u, u) \leq 1\}$. Since $((AA^*)^{-1}u, u) \leq \delta^{-1} \|u\|^2$, this ellipsoid contains in its interior a ball of radius $\sqrt{\delta}$.

Therefore, $\mu_\eta(A) \geq (\hat{\mu}_n(A))^{1/n} \to \delta^{n/2}$ as $n \to \infty$. Hence, $(\hat{\mu}_n(A))^{1/n} \geq \delta^{n/2} \forall n$. This contradicts the fact that $(\mu_\eta(A))^{1/n} \to 0$ as $n \to \infty$. Thus, $H_\delta$ is finite-dimensional for any $\delta > 0$. It is now easy to deduce that $AA^*$ is completely continuous, consequently, $A$ is also completely continuous.

§5. An upper bound for the Hausdorff dimension of attractors of semigroups corresponding to evolution equations

We consider semigroups of operators $\{S_t\}$ that have on the attractor a differential $S'_t$ in the weak sense (quasi-differential; see Definition 5.1). For many semigroups $\{S_t\}$ generated by evolution equations of the form $\frac{d}{dt}u = A(u)$ (as will be shown in §8) the differential $S'_t$ exists and is equal to the operator of shift along the trajectories of a linear equation of the form $\frac{d}{dt}v = L(t)v$, where $L(t) = A'(u(t))$ (see §8). To begin with we derive estimates of the constants $k$ for $(k, N)$-contracting non-linear operators $S_t$ in terms of their differentials. Then by using the results of §§3 and 4 we obtain an estimate of the dimension of the attractor of the semigroup.

Definition 5.1. Let $X$ be a subset of a Hilbert space $H$ and $S: X \to H$ a map defined on $X$. The map $S$ is said to be uniformly quasidifferentiable on $X$ in $H$ if for any $u \in X$ there is a linear operator $S'(u) : H \to H$ such that for any $u, v \in X$

$$\| S(u) - S(v) - S'(u)(u - v) \| \leq \gamma(\|u - v\|) \|u - v\|,$$

where $\gamma(z) \to 0$ as $z \to 0$ and $\gamma$ is independent of $u$ and $v$. The operator $S'(u)$ is said to be the quasidifferential of $S$ at $u$.

Lemma 5.1. Let $S: X \to H$ be a uniformly quasidifferentiable map on a compact set $X \subset H$. Suppose that the quasidifferentials $S'(u)$ are completely continuous and bounded in norm $\|S'(u)\| \leq M$, where $M$ is independent of $u \in X$. Let $N > 0$ be an integer and $\hat{\mu}_N(S'(u)) \leq k_0$, where $k_0$ is independent of $u \in X$. Then $S(2^Nk_0, N)$-contracts $X$ in $H$.

Proof. Without loss of generality we may assume that $M \geq k_0$ and $M \geq 1$. Let $u \in X$. We consider the quasidifferential $S'(u) = S'_t$. Since $S'_t$ is a completely continuous linear operator, by Proposition 4.3 to $S'_t$ there corresponds a projection operator $\pi^N = \pi^N(u)$, $\pi^N: H \to H^N$, and a bicylinder
\( \Lambda^N(\alpha_1, \ldots, \alpha_N) \) such that (3.3) is satisfied for \( x = 0 \) and \( S \) replaced by \( S' \). Since \( \|S'\| \leq M \), it follows from Proposition 4.2 that \( \alpha_i \leq M \). We now replace the lengths \( \alpha_i \) of the semi-axes of this bicylinder by \( \alpha'_i \) so that (3.3) remains valid, that is

\[
(5.2) \quad S'(B_r(0)) \subset r\Lambda^N(\alpha'_1, \ldots, \alpha'_N),
\]

and as before,

\[
(5.3) \quad \omega_N(\Lambda^N(\alpha'_1, \ldots, \alpha'_N)) \leq k_0,
\]

and also additionally

\[
(5.4) \quad \alpha'_N \geq m = k_0M^{1-N}.
\]

We set \( \alpha'_i = \alpha_i \) if \( \alpha_i \geq m \) and \( \alpha'_i = m \) if \( \alpha_i \leq m \) (\( i = 1, \ldots, N \)). Since \( \alpha'_i \geq \alpha_i \), (5.2) must be satisfied and (5.4) also holds. To verify (5.3) we remark that \( \alpha_i \leq \alpha_i \leq M \). Therefore,

\[
\omega_N(\Lambda^N(\alpha'_1, \ldots, \alpha'_N)) = \alpha'_1 \ldots \alpha'_N = \alpha_1 \ldots \alpha_{N-1}m \ldots m \leq M^{N-1}m^2 = k_0(k_0/M^N)^{q-1} \leq k_0.
\]

Let us now verify that the non-linear operator \( S(2^Nk_0, N) \)-contracts \( X \) in \( H \). Let \( u \in X \). We set \( \Phi(u, u_1) = S(u_1) - S(u) - S'(u)(u_1 - u) \). By (5.1),

\[
(5.5) \quad \| \Phi(u, u_1) \| \leq \gamma(R)R \quad \text{for} \quad \| u_1 - u \| \leq R, \ u_1, u \in X.
\]

To verify (3.3) it is enough to establish that \( S'(u)(u - u_1) + \Phi(u, u_1) \in \in 2r\Lambda^N(\alpha'_1, \ldots, \alpha'_N) \) for \( u, u_1 \in X \) and \( \| u - u_1 \| \leq R \) for sufficiently small \( r \).

By (5.2) and (5.4)

\[
\Phi(u, u_1) \in \gamma(r)r\Lambda^N(\alpha'_1, \ldots, \alpha'_N) = \gamma(r)rM^{N-1}k_0^{-1}\Lambda^N(\alpha'_1, \ldots, \alpha'_N).
\]

If \( r \) is so small that \( \gamma(r)M^{N-1}k_0^{-1} \leq 1 \), then \( \Phi(u, u_1) \in r\Lambda(\alpha'_1, \ldots, \alpha'_N) \), consequently, \( S'(u)(u - u_1) + \Phi(u, u_1) \in 2r\Lambda^N(\alpha'_1, \ldots, \alpha'_N) \). From this (3.3) follows with \( \alpha_i = 2\alpha'_i \). Taking (5.4) into account we find that

\[
\omega_N(2\Lambda^N(\alpha'_1, \ldots, \alpha'_N)) = 2^N\omega_N(\Lambda^N(\alpha'_1, \ldots, \alpha'_N)) \leq 2^Nk_0,
\]

and this completes the proof.

**Lemma 5.2.** Let \( S_1 \) and \( S_2 \) be uniformly differentiable operators on \( X \) in \( H \) with uniformly bounded quasidifferentials. Let \( S_1(X) \subset X \). Then the operator \( S_2 \circ S_1 \) is also uniformly quasidifferentiable on \( X \) in \( H \), and

\[
(S_2 \circ S_1)'(u) = S_2'(S_1(u)) \circ S_1(u).
\]

The proof is standard.

**Theorem 5.1** (see [42]). Suppose that \( X \) is a compact set in \( H \) and that the map \( S: X \to X \) is uniformly quasidifferentiable in \( H \) on \( X \) and such that the quasidifferentials \( S'(u) \) are completely continuous and uniformly bounded on \( X \) and that there is an \( N \) such that \( \mu_N(S'(u)) \leq k < 1 \), where \( k \) is independent of \( u \in X \). Let \( X \) be invariant under \( S \), \( S(X) = X \). Then the Hausdorff dimension of \( X \) in \( H \) does not exceed \( N : \dim X \leq N \).
Proof. By Lemma 5.2 the map \( S' \) is quasidifferentiable on \( X \) and the \( (S')' \) are uniformly bounded. It follows from (4.10) that 
\[
\hat{\mu}_N((S')') \leq \prod_{1}^{j} \mu_N(S') \leq k^j.
\]
Therefore, according to Lemma 5.1, \( S' \) is \( (2^Nk^j, N) \)-contracting on \( X \) in \( H \).
Since \( S(X) = X \), also \( S'(X) = X \). Choosing \( j \) so that \( 2^Nk^j < 1/C_N^0 \), where \( C_N^0 \) is the constant in Theorem 3.1 and applying this theorem to \( S' \), we find that \( \dim X \leq N \).

Remark 5.1. The assertion of Theorem 5.1 can be generalized to non-integral \( N \) (see [42]). The assumption that the \( S'(u) \) are completely continuous is not essential (see [42] and also the proof of Proposition 4.4).

Theorem 5.2. Let \( \{S_t\}, t \geq 0, \) be a semigroup of maps \( S_t : X \to X \), where \( X \subset H \) is compact and \( S_tX = X \) for all \( t \geq 0 \), and suppose that the operators \( S_t \) are uniformly quasidifferentiable in \( H \) on \( X \). Suppose also that for each \( u \in X \) the quasidifferential \( S'_t(u) \) is equal to \( G_t = G_t(u) \), the shift operator corresponding to an equation of the form (4.2), \( \partial_t v = L(t)v \), where the operator \( L(t) = L(t, u) \) satisfies Condition 4.1 and also (4.11):
\[
(L(t)v, v) \leq (-v(t)Q + h(t)I)v, v).
\]
Finally, suppose that the functions \( v(t) = v(t, u) \) and \( h(t) = h(t, u) \) in this condition satisfy the estimates
\[
\begin{align*}
\nu_t(u) &= \int_0^t \nu(\tau, u) \, d\tau \geq \nu_t^0 > 0, \\
h_t(u) &= \int_0^t h(\tau, u) \, d\tau \leq h_t^0 \quad \forall t \geq 0,
\end{align*}
\]
where \( \nu_t^0 \) and \( h_t^0 \) are independent of \( u \in X \). Let \(-\nu_t^0 \text{Tr}_N(Q) + Nh_t^0 < 0\). Then the Hausdorff dimension of \( X \) in \( H \) does not exceed \( N \):
\[
\dim X \leq N.
\]
If \( \eta(n), n > 0 \) is an increasing function such that
\[
\text{Tr}_N(Q)/n \geq \eta(n + 1) \quad \forall n \in \mathbb{N},
\]
then
\[
\dim X \leq \eta^{-1}(h_t^0/\nu_t^0),
\]
where \( \eta^{-1} \) is the inverse function of \( \eta \).

Proof. According to Theorem 4.2, it follows from
\[
-\nu_t^0 \text{Tr}_N(Q) + Nh_t^0 < 0
\]
that \( \hat{\mu}_N(G_t) \leq k < 1 \), consequently, \( \hat{\mu}_N(S'_t(u)) \leq k < 1 \) for all \( u \in X \). The complete continuity of \( G_t \) follows from (4.12): \( (\hat{\mu}_N(G_t))^{1/n} \to 0 \) as \( n \to \infty \), and Proposition 4.4. Using Theorem 5.1 we can now conclude at once that \( \dim X \leq N \). If (5.7) holds, then for (5.9) to hold it is sufficient that
\[
\text{Tr}_N(Q)/N \geq \eta(N + 1) > h_t^0/\nu_t^0,
\]
that is, \( N + 1 > \eta^{-1}(h_t^0/\nu_t^0) \). Hence we can choose \( N \) as the integral part of \( \eta^{-1}(h_t^0/\nu_t^0) \). Since \( \dim X \leq N \), this yields (5.8).
Remark 5.2. Using the concept of a flattening map (see §3) we can derive an estimate for the dimension similar to (5.8) (see [46]):

\[
\dim X \leq 2\eta^{-1} \left((2\kappa_+^2 + q_0)/\nu_0^2\right),
\]

where \(q_0\) is an absolute constant. Since the estimates for \(\dim X\) in specific problems based on (5.8) or (5.10) were deduced in the limit \(t \to \infty\), and since \(\nu_0^2 \to \infty\) as \(t \to \infty\), the value of \(q_0\) is irrelevant.

§6. A lower bound for the dimension of an attractor

If a semigroup \(\{S_t\}\) has a stationary point \(z\) and the operators \(S_t\) are sufficiently smooth in a neighbourhood of \(z\), then a so-called local unstable invariant manifold \(M(z)\) passes through \(z\). As will be shown below, this manifold is entirely contained within the attractor \(\mathcal{A}\) of \(\{S_t\}\). Therefore the dimension of \(M(z)\) can serve as a lower bound for the dimension of \(\mathcal{A}\).

Definition 6.1. Let \(z\) be a fixed point of a map \(S: O \to E\), where \(O\) is a neighbourhood of \(z\). Let \(\Lambda \subset O\) be a bounded set containing \(z\). A local unstable manifold \(M(z, \Lambda, S, r)\) where \(r > 1\) is the set of \(x \in \Lambda\) such that there is a sequence \(\{x_n, n = 0, 1, 2, \ldots\}\), \(x_n \in \Lambda\), satisfying the conditions: 1) \(x_0 = x\); 2) \(S^i x_k + j = x_i (i, j = 0, 1, \ldots)\); 3) \(\|x_n - z\| \leq C r^{-n}\) as \(n \to \infty\), \(C = C(\{x_n\})\).

Proposition 6.1. Let \(\{S_t\}\), \(S_t: E \to E\), \(t \geq 0\), be a semigroup having a maximal attractor \(\mathcal{A}\) and let \(z\) be a fixed point of \(S_t\) for a given \(t > 0\): \(S_t z = z\). Then \(M(z, \Lambda, S_t, r) \subset \mathcal{A}\).

(6.1) \(S_{tk}(M(z, \Lambda, S_t, r)) \supset M(z, \Lambda, S_t, r)\).

For if \(x \in M\), then by 2) \(x = x_0 = S_{tk} x_k\). We observe that \(x_k \in M(z, \Lambda, S_t, r)\), since the sequence \(y_i = x_{k+i} (i = 0, 1, \ldots)\) satisfies Definition 6.1, 1)-3). This proves (6.1). Since \(M(z, \Lambda, S_t, r) \subset \Lambda\) and \(\Lambda\) is bounded, we find that \(M(z, \Lambda, S_t, r)\) is bounded. From the fact that \(\mathcal{A}\) is attracting it follows that

\[
\text{dist}(S_{tk}(M(z, \Lambda, S_t, r)), \mathcal{A}) \to 0 \quad \text{as} \quad k \to \infty.
\]

Hence and from (6.1) it follows that \(\text{dist}(M(z, \Lambda, S_t, r), \mathcal{A}) = 0\), and since \(\mathcal{A}\) is closed, \(M(z, \Lambda, S_t, r) \subset \mathcal{A}\).

Corollary 6.1. Proposition 6.1 leads to the following estimate of the Hausdorff dimension of \(\mathcal{A}\):

\[
\dim M(z, \Lambda, S_t, r) \leq \dim \mathcal{A}.
\]

Theorem 6.1. Let \(z\) be a fixed point of \(S\) and let \(S\) be uniformly differentiable in a neighbourhood \(O\) of \(z\) and such that its differential \(S'(u)\) satisfies the Hölder condition

\[
\|S'(u_1) - S'(u_2)\| \leq C \|u_1 - u_2\|^\alpha \quad \forall u_1, u_2 \in O \quad (0 < \alpha < 1).
\]
Suppose that the spectrum of $S'(z)$ consists of two disjoint closed sets $\sigma_-$ and $\sigma_+$ where $\sigma_+$ is finite and

$$\sigma_- \subset \{ \lambda \in \mathbb{C} : |\lambda| \leq 1 \}, \quad \sigma_+ \subset \{ \lambda \in \mathbb{C} : |\lambda| > 1 \}.$$ 

Let $E_-$ and $E_+$ be, respectively, the invariant subspaces corresponding to $\sigma_-$ and $\sigma_+$ so that $E_+$ is finite-dimensional. Then there are sets $\lambda = \Omega_- \times \Omega_+ + z$, $\Omega_- \subset E_-$, $\Omega_+ \subset E_+$ and a differentiable map $g : \Omega_+ \to \Omega_-$ such that $g'$ satisfies the Hölder condition with exponent $\alpha$ and

$$(6.4) \quad M(z, \Lambda, S, r) = \{ u \in E : u = z + u_+ + g(u_+), \ u_+ \in \Omega_+ \},$$

where $r = 1 + \epsilon/2$ and $\epsilon > 0$ is sufficiently small, $g(0) = 0$, $g'(0) = 0$.

Theorem 6.1 is a consequence of a theorem proved in [19] (see [26]). The sets $\Omega_-$ and $\Omega_+$ are defined as follows: $\Omega_- = \{ u \in E_- : \| u \| \leq \rho \}$, $\Omega_+ = \{ u \in E_+ : \| u \|_+ \leq \rho \}$, where $\| \|$ and $\| \|_+$ are norms in $E_-$ and $E_+$ equivalent to the initial norm in $E$. These norms are chosen in such a way that $\| (S'(z))^{-1} \|_+ < 1 - \epsilon$ and $\| S'(z) \|_{E_- \rightarrow E_-} < 1 - \epsilon$, $\epsilon > 0$ (for the construction of such norms, see, for example, [4] and [26]).

**Corollary 6.2.** Let $z$ be a fixed point of a semigroup $\{S_t\}$ having an attractor $\mathcal{A}$. Suppose that for $t > 0$ the operator $S = S_t$ satisfies the hypotheses of Theorem 6.1 in some neighbourhood $O$ of $z$. Then the Hausdorff dimension of $\mathcal{A}$ satisfies

$$(6.5) \quad \dim \mathcal{A} \geq \dim E_+.$$ 

For under the conditions of the corollary, $\dim E_+ = \dim M(z, \Lambda, S, r)$, since according to the theorem the neighbourhood $\Omega_+$ of the origin in $E_+$ is mapped diffeomorphically by the function $u_+ \to z + u_+ + g(u_+)$ onto $M(z, \Lambda, S, r)$. This together with (6.2) yields (6.5).

In what follows we consider the case when the semigroup $\{S_t\}$ is generated by an evolution equation $\partial_t u = A(u)$ and its differential $S'_t$ is equal to the shift operator $G_t$ generated by an equation of the form $\partial_t v = A'(u(t))v$, $G_t : v(0) \to v(t)$. In this case, if $u(t) = z$ is a fixed point of $S_t$ for all $t$, then the operator $A'(u(t)) = A'(z)$ is independent of $t$, and $S'_t(z)$ is generated by the equation $\partial_t v = A'(z)v$.

**Theorem 6.2.** Let $S_t : E \to E$ be a semigroup, $z$ a fixed point of $\{S_t\}$, and suppose that the $S_t$ satisfy the conditions of Theorem 6.1 and that the differential $S'_t(z)$ is equal to the shift operator $G_t$ along the trajectories of a linear equation of the form

$$(6.6) \quad \partial_t v = L v.$$ 

Suppose that $L$ has a finite-dimensional invariant subspace $E_+^0$ with a basis consisting of eigenvectors and the adjoint vectors of $L$ corresponding to the eigenvalues $\mu_j$ ($j = 1, \ldots, k$) with $\text{Re} \ \mu_j > 0$. Suppose that $\{S_t\}$ has a
maximal attractor $\mathcal{A}$. Then

$$\dim \mathcal{A} \geq \dim E_0^o.$$  

Proof. Clearly, the subspace $E_0^o$ is invariant under the operator $G_t$ corresponding to (6.6). Here the eigenvalues of $G_t$ on $E_0^o$ are in modulus greater than 1. Therefore, $E_0^o \subset E_+$. Using Theorem 6.1 and Corollary 6.2, we now obtain (6.7) from (6.5).

Remark 6.1. The solutions $v(t)$ of (6.6) with initial data in $E_0^o$ clearly tend to infinity (exponentially) as $t \to +\infty$. This is why the vectors in $E_0^o$ are called the unstable directions at a stationary point $z$.

Remark 6.2. In all our examples the operators $S_t'(z), t > 0$, are completely continuous. Therefore, their spectrum is discrete, and, evidently, the spectrum of $S_t'(z)$ splits into closed sets $\sigma_+$ and $\sigma_-$, where the corresponding subspace $E_+$ is finite-dimensional.

§7. Differentiability of shift operators

In this section we prove that the shift operators $S_t: u(0) \to u(t)$ corresponding to evolution equations of the form

$$\frac{d}{dt} u(t) = A(u(t)), \quad u(0) = u_0,$$

which were studied in §2, are differentiable and quasidifferentiable. We require in what follows that the $S_t$ are quasidifferentiable on the attractor $\mathcal{A}$ in $L_2(\Omega)$, so as to apply Theorem 5.2 to obtain an upper bound for $\dim \mathcal{A}$.

unstable manifold at the fixed points of $S_t$, which enables us to give a lower bound for $\dim \mathcal{A}$ by means of Theorem 6.2.

As a rule, the differentiability of $S_t$ is proved the easier the smoother the functions in $E$ on which $S_t$ acts. For example, it is especially simple to prove the differentiability of $S_t$ when $E = C^\beta(\Omega)$, where $\beta$ is sufficiently large. However, various a priori estimates used in the proof of the existence of an attractor and in the estimates of its dimension are usually more readily established in spaces of type $L_2(\Omega)$ or $H_1(\Omega)$. Therefore, we prove the differentiability and quasidifferentiability in the latter spaces, although this sometimes imposes additional restrictions on the form of the equations (7.1).

Before dealing with individual examples we give a sketch of the general method of proof of uniform differentiability and quasidifferentiability of $S_t$ acting on a Banach space $E$. Let $B$ be a subset of $E$, $u_0 \in B$, and let $u(t)$ be the solution of (7.1) satisfying the initial condition $u(0) = u_0$. By differentiating (7.1) formally with respect to the initial data we obtain the variational linear equation

$$\frac{d}{dt} v(t) = A'(u(t))v(t), \quad v(0) = v_0,$$

which has the form (4.2). To (7.2) there corresponds a family of linear
shift operators $G_t = G_t(u_0), G_t : E \to E, G_t v_0 = v(t)$. We are required to prove that $G_t(u_0)$ is the differential or quasidifferential of $S_t$ at $u_0$. For this purpose we consider the difference

$$w(t) = S_t(u_0 + v_0) - S_t(u_0) - G_t(u_0)v_0, \quad v_0 \in E.$$  

(7.3)

The operator $G_t(u_0)$ coincides with the differential $S'_t(u_0)$, and $S_t(u)$ is uniformly differentiable on $B$ if for small $\|v_0\|_E$ and a given $t$

$$\|w(t)\|_E \leq C \|v_0\|_E^{1+\delta}, \quad \delta > 0,$$  

(7.4)

where $C$ is independent of $u_0 \in B$ and depends only on $B$. (In the case of quasidifferentiability on $B$ it is sufficient to derive (7.4) for $u_0 \in B$ and such $v_0$ for which $u_0 + v_0 \in B$.) To prove (7.4) we use the fact that $u^t(\cdot) = S_t(u_0 + v_0)$ is a solution of (7.1) with the initial condition $u_0 + v_0$, the function $u^t(\cdot) = S_t u_0$ is the solution of (7.1) with the initial condition $u(0) = u_0$, and $v(t)$ is the solution of (7.2). We rewrite (7.3) in the form

$$w(t) = u_t(t) - u(t) - v(t).$$  

(7.5)

Subtracting from (7.1) for $u_t(t)$ the equation (7.1) for $u(t)$ and (7.2) we obtain an equation for $w(t)$ of the form

$$\partial_t w(t) = A(u(t) + v(t) + w(t)) - A(u(t)) - A'(u(t))v(t),$$  

(7.6)

$$w(0) = 0.$$  

We rewrite this equation in the form

$$\partial_t w(t) - A'(u(t))w(t) = A_0(u, v, w), \quad w(0) = 0,$$  

(7.7)

where

$$A_0(u, v, w) = A(u(t) + v(t) + w(t)) - A(u(t)) - A'(u(t))(v(t) + w(t)).$$  

(7.8)

Obviously, $A_0(u, v, w)$ is of higher than the first order in $v$ and $w$.

When we consider actual examples of (7.1) with initial data in $B$, we can in many cases obtain the estimate

$$\|w(t)\|_E \leq \int_0^t \phi(\tau) \|w(\tau)\|_E^2 d\tau + C_0 \|v_0\|_E^{2+\gamma}, \quad \phi(\tau) \geq 0.$$  

(7.9)

Proposition 7.1. Suppose that (7.9) holds for $0 < t < T$ and that

$$\int_0^T \phi(\tau) d\tau \leq M.$$  

(7.10)

Then for $0 < t < T$

$$\|w(t)\|_E \leq C_0 e^M \|v_0\|^{2+\gamma}.$$  

(7.11)

Proof. By Gronwall's inequality we obtain from (7.9)

$$\|w(t)\|_E^2 \leq C_0 \|v_0\|^{2+\gamma} \exp \left( \int_0^t \phi(\tau) d\tau \right),$$  

from which (7.11) follows at once by (7.10).
By Proposition 7.1, having (7.9) for a solution of (7.7), where \( \gamma > 0 \), we obtain (7.4). Hence the operator \( G_t(u_0) \) is the differential (or quasidifferential) of \( S_t \) at \( u_0 \). If the constants \( C_0, M, \) and \( \gamma \) in (7.9) and (7.10) are independent of \( u_0 \in B \) and depend only on \( B \), then \( S_t \) is uniformly differentiable (quasidifferentiable) on \( B \).

We now remark that to verify the Hölder condition (6.3) we consider the difference \( v_1(t) - v_2(t) = S'_t(u_0) v_0 - S'_t(u_0) v_0 \). It is a solution of the equation obtained by subtracting the corresponding equations (7.2):

\[
(7.12) \quad \partial_t (v_1 - v_2) = A'(u_2(t))(v_1 - v_2) = (A'(u_2(t)) - A'(u_2(t))) v_2,
\]

\[
v_1(0) = v_2(0) = v_0.
\]

To obtain (6.3) we use the continuous dependence of the solution of the linear equation (7.12) on its right-hand side in the corresponding norms and also estimates for \( \| u_1(t) - u_2(t) \| \) in terms of \( \| u_0_1 - u_0_2 \| \).

To derive inequalities in the form (7.9) and (6.3) we need an upper bound for the right-hand sides of (7.7) and (7.12) in the corresponding norms. If the norms in question are integral (such as in \( W^p \)), we frequently use the following standard inequalities: the Hölder inequality

\[
\| u \|_{L^p} \leq C_{\| u \|_{L^q} \cdot \| u \|_{L^{p'}}} \quad \text{for} \quad 0 \leq q \leq 1,
\]

the Gagliardo-Nirenberg inequality (see, for example [21])

\[
\| u \|_{L^p} \leq C_\| u \|_{L^q} \cdot \| u \|_{L^{q'}} \quad \text{for} \quad 0 \leq q \leq 1,
\]

where \( \lambda = n/2 - n/p \) and also the interpolation inequality

\[
\| u \|_{L^q} \leq C_{\| u \|_{L^{q'}}} \quad \text{for} \quad 0 \leq q \leq 1.
\]

Frequently one uses the elementary numerical inequality

\[
(7.15) \quad ab \leq e^{a q/q + e^{1-q' b'/q' q' q}}, \quad 1/q + 1/q' = 1, \quad q > 1, \quad a, b > 0.
\]

We now pass on to the discussion of the equations in §2.

For the equation with monotonic principal part in Example 2.1 in the case when the principal part contains non-linear terms we cannot prove the differentiability of the operator \( S_t : L_2(T^n) \to L_2(T^n) \); this is apparently an inherent problem.

We now turn to Example 2.2 and consider (2.11). The variational equation (7.2) corresponding to (2.11), which is obtained by formal differentiation of (2.11) with respect to \( u \), has the form

\[
(7.16) \quad \partial_t v = \sum [a_{ij}(x, u) \partial_i \partial_j v + (a_{ij})_u \partial_i \partial_j u] v + \sum b_{ij} u \partial_i v + b_{ij} v - (a_0)_{u} v = A'(u) v, \quad v \big|_{\partial_2} = 0.
\]
**Theorem 7.1.** Let $B \subset E$, where $E$ is defined by (2.16) and $B$ is bounded in $E$. Let $u(t) = S_t u_0$, $u_0 \in B$ and let $\{S_t\}$ be the semigroup corresponding to (2.11). Then for any $v_0 \in E$ the problem (7.16) has a unique solution $v(t) \in V^{2+\alpha}(0, T)$ for all $T > 0$ satisfying the initial condition $v(0) = v_0$. The correspondence $v_0 \to v(t)$ determines a linear bounded operator $G_t$, which is the Fréchet differential of $S_t : E \to E$ at $u_0$. The differential $S'_t(u_0) = G_t(u_0)$ satisfies the Hölder condition (6.3) on $B$.

If $u_0 \in \mathfrak{M}$, where $\mathfrak{M}$ is an attractor for $\{S_t\}$, then $G_t$ is the quasidifferential of $S_t$ in $L_2(\Omega)$ on $\mathfrak{M}$ at $u_0$. Here $S_t$ is uniformly quasidifferentiable on $\mathfrak{M}$ in $L_2(\Omega)$.

**Proof.** Since by Theorem 2.4 the function $u(t)$ is bounded in $V^{2+\alpha}(0, T)$, the coefficients of the linear equation (7.16) belong to $V^\alpha(0, T)$ and by standard theorems on the solubility of such equations there is a unique solution $v(t)$ for $v(0) = v_0$ and

\[
\|v(t)\|_{V^{2+\alpha}} \leq C_T \|v_0\|_E, \quad V^{2+\alpha} = V^{2+\alpha}(0, T).
\]

And so the operator $G_t$ is well defined.

To prove the differentiability of $S_t$ we consider the equation (7.7) corresponding to (2.11). Its right-hand side $A_0(u, v, w)$ is a function of $u, v,$ and $w$ and their derivatives and is of second order of smallness in $w$ and $v$. Simple but somewhat tedious computations show that

\[
\|A_0 (u, v, w)\|_{V^\alpha} \leq C_1 (\|v\|_{V^{2+\alpha}}^2 + \|w\|_{V^{2+\alpha}}^2).
\]

Since $u, v, w \in V^{2+\alpha}$, the coefficients on the right-hand side of (7.7) belong to $V^\alpha$. Therefore, from standard properties of linear parabolic equations we deduce that

\[
\|w\|_{V^{2+\alpha}} \leq C \|A_0\|_{V^\alpha} \leq C_2 (\|v\|_{V^{2+\alpha}}^2 + \|w\|_{V^{2+\alpha}}^2).
\]

Hence, bearing in mind (7.17) and the fact that $v$ and $w$ are small, we obtain (7.4) with $\delta = 1$. Similarly, considering (7.12), we can derive the estimate (6.3) with $\alpha = 1$.

We now come to the proof that $S_t$ is quasidifferentiable on $\mathfrak{M}$ in $L_2(\Omega)$. Since by Theorem 2.6 $\mathfrak{M}$ is bounded in $C^{3+\delta}(\Omega)$, the functions $u_t(t) = S_t u_{t_0}$ and $u_2(t) = S_t (u_{t_0} + v_0)$, $u_{t_0} + v_0 \in \mathfrak{M}$ are bounded in $V^{2+\alpha}$, and by (7.17) $v(t)$ is bounded in $V^{2+\alpha}$. (The norms of $u_{t_1}, u_{t_2}$, and $v$ in $V^{2+\alpha}$ depend only on $T$ and $\mathfrak{M}$.) Thus, the coefficients of (7.7) are bounded in $V^\alpha$ and the functions $u, v,$ and $w$ on the right-hand side of $A_0(u, v, w)$ are bounded in $V^{2+\alpha}$ by a constant $R$. For simplicity we restrict ourselves to the case when the coefficients $a_{ij}$ in (2.11) are independent of $u$ and the function $b(x, u, \nabla u)$ depends linearly on $\nabla u$ and is independent of $u$.

In this case, $A_0$ takes the form

\[
A_0 (u, v, w) = -(a_0 (u + v + w) - a_0 (u) - a'_0 (u) (v + w)).
\]
Multiplying (7.7), where $A'$ is defined by (7.16), by $2w(t)$ and integrating with respect to $x$ and $t$, we obtain

$$
(7.20) \quad \| w(t) \|^2 + \int_0^t (2\mu_0 \| w \|^4 - C \| w \| - C \| w \|^2) \, d\tau \leq
$$

$$
\leq 2 \int_0^t \| (A_0 (u, v, w), w) \| d\tau, \quad C = C (R).
$$

(Here $(\cdot, \cdot)$ is the scalar product in $L_2(\Omega)$.) By (7.19),

$$
(7.21) \quad |(A_0, w)| \leq C \int_\Omega |v + w|^2 |dx| \leq C_2 \int_\Omega |v|^{2+\gamma_0} |dx| + C_2 \| w \|^2, \quad 0 < \gamma_0 \leq 1.
$$

Here we have used the fact that $w$, $u$, and $v$ are bounded in $C^{2+\alpha}$ and hence also in $C(\Omega)$. This is a typical feature of a proof of quasidifferentiability: $v$ and $w$ are small in $L_2$, but bounded in $C^{2+\alpha}$. We choose $\gamma_0$ sufficiently small.

To estimate the last term but one in (7.21), we make use of (7.14) with $l = 0$ and $p = 2+\gamma_0$ and we obtain from (7.21)

$$
(7.22) \quad |(A_0, w)| \leq C_3 \| w \|^2 + C_4 \| v \|^2 + \| v \|^{2+\gamma_0 - \beta},
$$

where $\beta < 2$, since $\gamma_0$ is small. From (7.20) and (7.22) we deduce that

$$
(7.23) \quad \| w(t) \|^2 + \mu_0 \int_0^t \| w(\tau) \|^4 d\tau - C_5 \int_0^t \| w(\tau) \|^2 d\tau \leq
$$

$$
\leq C \int_\Omega \| v \|^2 + \| v \|^{2+\gamma_0 - \beta} \int_\Omega \| v \| \beta \, d\tau.
$$

As is well known, the solutions of the linear parabolic equation (7.16) with sufficiently regular coefficients (this condition is satisfied in our case, since $u_1$, $u_2$, and $v$ are smooth) satisfy

$$
(7.24) \quad \sup_{0 \leq t \leq T} \| v(\tau) \|^2 + \mu_0 \int_0^t \| v(\tau) \|^4 d\tau \leq C \| v(0) \|^2, \quad 0 \leq t \leq T.
$$

From (7.23) and (7.24) we derive (7.9), where $\gamma = \gamma_0 > 0$, $\varphi(\tau) = C_5$, and $E = L_2(\Omega)$. The uniform quasidifferentiability of $S_t$ on $\mathcal{W}$ follows from (7.9) in the standard way.

The case when (2.11) contains non-linear terms in the higher derivatives can be treated similarly, although certain complications may arise (we do not require this case in the present paper).

Let us now consider the semigroup $\{S_t\}$ in Example 2.3. The variational equation (7.2) corresponding to (2.21) has the form

$$
(7.25) \quad \partial_t v = a \Delta v + \sum_i \partial_i ((b_i)' u) v - f_i' v + \lambda v \equiv A' (u(t)) v, \quad v |_{\partial \Omega} = 0.
$$

Here $(b_i)' u$ and $f_i'$ are the matrices of the derivatives of the vector-valued functions $b_i$ and $f$ with respect to $u$. We require that in addition to (2.22),
(2.23), and (2.24) also
\begin{equation}
\begin{aligned}
|b_t(u+z)-b_t(u)-(b_t(u))'_z z| & \leq C (1 + |u| + |z|)^{p_1/2}|z|^{1+\gamma_1/2}, \\
|f(u+z)-f(u)-f'_u(u) z| & \leq C (1 + |u| + |z|)^{p_1}|z|^{1+\gamma_1} \quad \forall x \in \Omega,
\end{aligned}
\end{equation}
where $C$ is independent of $u$, $z \in \mathbb{R}^m$, $x \in \Omega$, $\gamma_1 \in (0, 1]$, $\gamma_1$ is sufficiently small and
\begin{equation}
p_1 < +\infty \quad \text{for} \quad n = 2, \quad p_1 < 4/(n-2) \quad \text{for} \quad n > 2.
\end{equation}

**Theorem 7.2.** Let $B \subset (L_2(\Omega))^m$ be a set that is bounded in norm in $(H_1(\Omega))^m$, and $u(t) = S_t u_0$, $u_0 \in B$. Then for any $v_0 \in (L_2(\Omega))^m$ the equation (7.25) with the initial condition $v(0) = v_0$ has a unique solution $v(t)$ satisfying (7.24).

Let $\mathcal{A}$ be the attractor for $\{S_t\}$, $S_t : (L_2(\Omega))^m \to (L_2(\Omega))^m$, corresponding to (2.21), $u_0 \in \mathcal{A}$, and $S_t u_0 = u(t)$. Then the shift operator $G_t$, $G_t v_0 = v(t)$, corresponding to (7.25) is the quasidifferential of $S_t$ on $\mathcal{A}$ in $(L_2(\Omega))^m$ at $u_0$.

The operator $S_t$ for $t > 0$ is uniformly quasidifferentiable on $\mathcal{A}$ in $(L_2(\Omega))^m$.

**Proof.** The existence and uniqueness of the solution of (7.25) and also the estimate (7.24) are well known. By Theorem 2.9, the attractor $\mathcal{A}$ is bounded in $(H_1(\Omega))^m$ by a constant, which we denote by $R$.

To prove that $S_t$ is quasidifferentiable on $\mathcal{A}$ we consider (7.7), where $A(u(t))$ is the same as in (7.25). Multiplying (7.7) by $2w(t)$ and integrating with respect to $x$ and $t$, we can use the fact that $a_+ \geq \mu_0 I$ to derive (7.20), where $(,)$ is the scalar product in $(L_2(\Omega))^m$. By means of (7.26) we deduce
\begin{equation}
2|\langle A_0 (u, v, w), w \rangle| \leq C \int_\Omega [(1 + |u| + |v + w|)^{p_1/2}|v + w|^{1+\gamma_1/2} |\nabla w| +
\end{equation}
\begin{equation}
+ (1 + |u| + |v + w|)^{p_1}|v + w|^{1+\gamma_1}|w|] \, dx \leq \frac{H_0}{2} \|w\|_2^2 + C_2 \int_\Omega [(1 + |u| + |v + w|)^{p_1} (|v|^{2+\gamma_1} + |w|^{2+\gamma_1}) \, dx.
\end{equation}

Since $u(t) = S_t u_0 \in \mathcal{A}$, $u(t) = S_t (u_0 + v_0) = u + v + w \in \mathcal{A}$, and the set $\mathcal{A}$ is bounded in $H_1$, the functions $u(t)$ and $v(t) + w(t) = u(t) - u(t)$ are bounded in $H_1$ by a constant independent of $t$. We consider only the case $n > 2$, the case $n \leq 2$ is simpler. By the Sobolev embedding theorem, $H_1(\Omega) \subset L_{q_0}(\Omega)$, where $1/q_0 = 1/2 - 1/n$, therefore $u$ and $v + w$ are bounded in $L_{q_0}(\Omega)$ by a constant depending only on $R$. Using the Hölder inequality to estimate the integral on the right-hand side of (7.28), we obtain
\begin{equation}
2|\langle A_0, w \rangle| \leq \frac{H_0}{2} \|w\|_2^2 + C_2 \|\langle 1 + |u| + |v + w| \rangle\|_{L_{q_0}^\infty} (\|v\|_{L_{q_0}^{2+\gamma_1}} + \|w\|_{L_{q_0}^{2+\gamma_1}}) \leq \frac{H_0}{2} \|w\|_2^2 + C_3 (R) (\|v\|_{L_{q_0}^{2+\gamma_1}} + \|w\|_{L_{q_0}^{2+\gamma_1}}),
\end{equation}
where $q_1 = (2 + \gamma_1)q_0/(q_0 - p_1)$. 

\[A. V. Babin and M. I. Vishik\]
Estimating the last term in (7.29) by means of (7.14), we conclude that

\[(7.30) \quad |(A_0, w)| \leq \frac{\mu_0}{4} \| w \|_p^2 + C_4 (\| w \|_{2+\gamma_1} + \| v \|_{2+\gamma_1} + \| w \|_{2+\gamma_1}),\]

where \( \alpha = \frac{\gamma_1}{2} + p_1(n-2)/2 \). Evidently, \( \alpha < 2 \) for sufficiently small \( \gamma_1 \), by (7.27). Making use of (7.15) and of the fact that \( w \) is bounded in \( L_2 \), we deduce from (7.30) that

\[(7.31) \quad |(A_0, w)| \leq \frac{1}{4} \mu_0 \| w \|_1^2 + C_2 \| w \|_2^2 + C_4 \| v \|_{2+\gamma_1} + \| v \|_2.
\]

We can use (7.31) and (7.20) to derive (7.23), where \( \gamma_0 = \gamma_1 \) and \( \beta = \alpha \), and by (7.24) this yields (7.9), where \( \| \|_E = \| \| \) and \( \gamma = \gamma_1 \). It is obvious that the constants in (7.9), and hence also in (7.11), depend only on \( R \), which implies the uniform quasidifferentiability of \( S_t \) on \( \mathcal{A} \).

**Theorem 7.3.** Let \( \{ S_t \} \) be the semigroup corresponding to (2.21) and suppose that the conditions (2.22), (2.23), (2.24), and (7.26) are satisfied and that the functions \( b_t(x, u) \) are linear in \( u \) and also that

\[(7.32) \quad |f'_u (u_1) - f'_u (u_2)| \leq C ([|u_1| + |u_2| + 1]p_1 |u_1 - u_2|),\]

where \( p_1 \) satisfies (7.27), \( \gamma_1 \in (0, 1] \), and \( \gamma_1 \) is small. Then the operators \( S_t, t \geq 0, \) restricted to \( (H^0_1(\Omega))^m \) map \( (H^0_1(\Omega))^m \) into \( (H^0_1(\Omega))^m \). The operator \( S_t \) restricted to \( (H^0_1(\Omega))^m \) is uniformly differentiable on sets that are bounded in \( (H^0_1(\Omega))^m \), and the differential \( S'_t (u_0) \) is the shift operator \( G_t \) corresponding to (7.25). The operators \( S'_t : (H^0_1(\Omega))^m \to (H^0_1(\Omega))^m \) satisfy the Hölder condition (6.3).

The only difference is the fact that to estimate \( S_t \) we evaluate estimates in \( (H^0_1(\Omega))^m \) of the functions \( u(t) \) and \( w(t) \), the equation (2.21) and the corresponding equation (7.7) are multiplied by \( \Delta u \) and \( \Delta w \). The terms on the right-hand side of (7.7) can be estimated by means of (7.13) and (7.14) with \( l = 1 \). To derive (6.3), we consider the equation (7.12) corresponding to (2.21). When we estimate the right-hand side of this equation as for (7.7), we arrive at (6.3).

**§8. Estimates of the Hausdorff dimension of an attractor for a two-dimensional Navier-Stokes system**

In §8.1 we show that all the conditions of Theorem 5.2 are satisfied for a two-dimensional Navier-Stokes systems under zero boundary conditions. On the basis of this theorem we derive an upper bound for the Hausdorff dimension of the attractor: \( \dim \mathcal{A} \leq C v^{-4} \| f \|_{1,2}^2 \).

Under periodic boundary conditions on the period parallelepiped \( \Omega_0 = [0, 2\pi/\alpha_0] \times [0, 2\pi] \), where \( \alpha_0 \) is small, we show in §8.2 that the estimate (3) quoted in the Introduction holds. In §8.3 we derive a lower bound for the dimension of the attractor: \( \dim \mathcal{A} \geq C \alpha^{-6} \).
Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with sufficiently smooth boundary $\partial \Omega$. We consider the two-dimensional Navier-Stokes system

\begin{equation}
\partial_t u + u \cdot \nabla u + \nu \Delta u + \nabla p = g(x), \quad \nabla u = 0,
\end{equation}

where $(x_1, x_2) \in \Omega$, $t > 0$, $\nu > 0$, $u = (u_1(x, t), u_2(x, t))$, and $g(x) \in (L^2(\Omega))^2$.

Two boundary-value problems are studied. In the first problem adhesion conditions are given on $\partial \Omega$:

\begin{equation}
u \mathbf{n} \cdot \mathbf{u} = 0.
\end{equation}

The second boundary-value problem is periodic with the period parallelepiped $\Omega_0 = [0, 2\pi/\alpha_0] \times [0, 2\pi]$, $\alpha_0 > 0$,

\begin{equation}
\mathbf{u}(x_1 + 2\pi/\alpha_0, x_2 + 2\pi, t) = \mathbf{u}(x_1, x_2, t) \quad \forall t > 0, \quad x \in \mathbb{R}^2.
\end{equation}

Obviously, in this case the problem (8.1), (8.3) is equivalent to the corresponding problem on the torus $T^2(\alpha_0) = T^2$. In the periodic case it is assumed that the average values of $\mathbf{u}$ and $g$ on $T^2$ are zero:

\begin{equation}
\frac{1}{T^2} \int_{T^2} \mathbf{u}(x, t) \, dx = 0, \quad \frac{1}{T^2} \int_{T^2} g(x) \, dx = 0.
\end{equation}

1. We consider first the question of the existence of an attractor and of upper estimates for its Hausdorff dimension in the case (8.1), (8.2). The norm and scalar products in $(L^2(\Omega))^2$ are denoted by $\| \cdot \|$ and $(\cdot, \cdot)$; $(\nabla \mathbf{u}, \nabla \mathbf{u}) = \| \nabla \mathbf{u} \|^2$. We denote by $H$ or $V$ the closures in the norms $\| \cdot \|$ or $\| \cdot \|_1$, respectively, of the set

\begin{equation}
V_0 = \{ v \in (C_0^\infty(\Omega))^2 : \text{div} \, v = 0 \}.
\end{equation}

We denote by $\Pi$ the orthogonal projection in $(L^2(\Omega))^2$ onto $H$ and its various extensions. As usual, by projecting (8.1) onto $H$ we eliminate $\nabla p$ and obtain the equation

\begin{equation}
\partial_t \mathbf{u} + \nu \mathbf{L} \mathbf{u} + B(\mathbf{u}, \mathbf{u}) = \mathbf{f}, \quad \mathbf{L} = -\nabla \Delta,
\end{equation}

where $B(\mathbf{u}, \mathbf{v}) = \nu (u_1 \partial_1 v_1 + u_2 \partial_2 v_2)$, $f = \Pi g$. At $t = 0$ we impose the initial condition

\begin{equation}
\mathbf{u} \mid_{t=0} = \mathbf{u}_0(x), \quad u_0 \in H.
\end{equation}

As is well known (see [32], [24], and [25]), the problem (8.5), (8.6) has one and only one solution $u(x, t)$ belonging to $C([0, T]; H) \cap L_2(0, T; V)$. Thus, the semigroup $\{ S_t, t \geq 0 \}$ whose action is given by

\begin{equation}
S_t u_0 = u(t), \quad S_t : H \to H \quad \forall t \geq 0,
\end{equation}

is well defined; here $u(t)$ is the solution of (8.5), (8.6).

We quote the standard a priori estimates for the solutions $u(t)$ indicated above. Taking the scalar product of (8.5) with $u(t)$ we obtain

\begin{equation}
\frac{1}{2} \partial_t \| u \|^2 + \nu \| u \|_2^2 + (B(u, u), u) = (f, u).
\end{equation}
Attractors of partial differential evolution equations and estimates of their dimension

Since $(B(u, u), u) = 0$, we deduce

$$\partial_t \|u\|^2 + 2\nu \|u\|_1 \leq 2 \|f\|_{-1} \|u\|_1 \leq \frac{1}{\nu} \|f\|_{-1} + \nu \|u\|_1^2.$$  

(Here $\| \cdot \|$ denotes the norm in the scale of spaces $H_\varepsilon$ generated by $L_0$. Obviously, $H_1 = V$.) Since $\|u\|_1^2 \geq \gamma_1 \|u\|^2$, where $\gamma_1$ is the first eigenvalue of $L_0$, (8.8) leads to the differential inequality

$$\partial_t \|u\|^2 + \nu \gamma_1 \|u\|_1^2 \leq \frac{1}{\nu} \|f\|_{-1}^2,$$

from which we derive the estimate

$$\|u(t)\|^2 \leq (\|u(0)\|^2 - \nu^{-2} \gamma_1^{-1} \|f\|_{-1}^2) e^{-\nu t} + \nu^{-2} \gamma_1^{-1} \|f\|_{-1}^2.$$  

Integrating (8.8) with respect to $t$ we obtain

$$\|u(T)\|^2 + \nu \int_0^T \|u(t)\|_1^2 \, dt \leq T \nu^{-1} \|f\|_{-1}^2 + \|u(0)\|^2.$$  

**Theorem 8.1.** The semigroup $\{S_t\}$ defined by (8.7) has the properties:

1) for any $t > 0$ the operators $S_t : H \to H$ are continuous;

2) $\{S_t\}$ is uniformly bounded in $H$;

3) there is an absorbing set bounded in $H$ corresponding to $\{S_t\}$;

4) the operators $S_t$ for $t > 0$ map bounded sets in $H_1$ into bounded sets in $H_1$;

5) $\{S_t\}$ has a maximal attractor $\mathbb{A}$ in $H$ and $\mathbb{A}$ is bounded in $H_1$;

6) the trajectories $u(t)$ for $u = u(0) \in \mathbb{A}$ are subject to the estimates

$$\|u(t)\|^2 \leq \|f\|_{-1}^2 + \nu^{-2} \gamma_1^{-1} \|f\|_{-1}^2.$$  

Proof. The proof of 1) follows from the standard theorem on the continuous dependence of $u(t)$ on the initial data (see [32], [24], and [25]). (In Lemma 8.2 we prove that $u(t)$ is differentiable with respect to $u_0$, which implies the continuity.) 2) follows from (8.9). For if $\|u(0)\| \leq R$, then by (8.9) $\|u(t)\| \leq C(R, \|f\|_{-1})$. 3) also follows from (8.9), since $\|u(t)\| \leq C(t) \|f\|_{-1} \leq R$ and $t \geq t_0(R)$. A proof of 4) can be found, for example, in [11]. It is based on (8.10) and also on an estimate derived from (8.5) by taking the scalar product with $L_0 u$. The existence of an attractor $\mathbb{A}$ for $\{S_t\}$ follows from Theorem 1.2, since all its hypotheses are fulfilled by 1)–4). Since $S_t \mathbb{A} = \mathbb{A}$ for all $t > 0$ and $\mathbb{A}$ is bounded in $H$, it follows from 4) that it is also bounded in $H_1$. It remains to prove 6). The estimate (8.11) follows from (8.9). For, if $u \in \mathbb{A}$, then for any $t > 0$ there is a $v \in \mathbb{A}$ such that $S_t v = u$. Consequently, (8.9) holds for $u$ with $u(t)$ replaced by $u$ and $u(0)$ by $v$. Since $\mathbb{A}$ is bounded in $H$ and $t$ can be arbitrarily large, (8.9) implies (8.11). The estimate (8.12) follows from (8.11) and (8.10).
Let us write down the variational equation corresponding to (8.5):

$$\partial_t \nu = v(\Pi \Delta u - B(u(t), \nu) - B(\nu, u(t)) = A'(u(t))\nu. \tag{8.13}$$

**Lemma 8.1.** Suppose that $u(t) \in C([0, T]; H)$ and that $u(t)$ is bounded in $H_1 = V$ uniformly in $t \in [0, T]$. Then the operator $L(t) = A'(u(t))$ satisfies Condition 4.1, where $Q = -\Pi \Delta = L_0, H_1 = V$ and $H_{-1} = V'$.

**Proof.** We consider the bilinear form

$$\langle L(t)\nu, w \rangle = \nu(\Pi \Delta \nu, \nu) - (B(u, \nu), w) - (B(\nu, u), w) = \nu(\Pi \Delta \nu, w) + (B(u, \nu), \nu) + (B(\nu, u), \nu), \quad \nu, w \in V. \tag{8.14}$$

We have used the fact that

$$\langle B(u, \nu), w \rangle = -(B(u, w), \nu). \tag{8.15}$$

Bearing in mind the standard inequality (see [32] and [24])

$$\|u\|_{L^2} \leq C \|u\|_1 \|u\|, \tag{8.16}$$

we obtain

$$\|\langle B(u, v), u \rangle\|_{L^2} \leq C \|u\|_1 \|u\|_1 \|u\|_1 \|u\|_1 \leq C \|u\|_1 \|u\|_1 \|u\|_1 \|u\|_1 \leq C \|u\|_1 \|u\|_1 \|u\|_1 \|u\|_1 \|u\|_1. \tag{8.17}$$

This together with (8.14) yields

$$\|\langle L(t)\nu, w \rangle\| \leq C \|u\|_1 \|u\|_1 \|u\|_1 \|u\|_1 \|u\|_1 \leq C \|u\|_1 \|u\|_1 \|u\|_1 \|u\|_1 \|u\|_1 \|u\|_1. \tag{8.18}$$

This implies (4.3). Setting $w = \nu$ in (8.14) and using (8.15) and (8.17), we find that

$$\|\langle L(t)\nu, \nu \rangle\| = -\nu \|\nu\|_1^2 - (B(\nu, u), \nu) \leq -\nu \|\nu\|_1^2 - C_2 \|u\|_1 \|\nu\|_1 \|\nu\|_1 \|\nu\|_1 \|\nu\|_1 \|\nu\|_1 \|\nu\|_1. \tag{8.19}$$

This implies (4.4). To verify (4.5) we set $v = e_i$ and $w = e_j$ in (8.14), where $e_i$ and $e_j$ are eigenvectors of $\Pi \Delta$. Since the boundary $\partial \Omega$ is smooth, $e_i$ and $e_j$ are sufficiently smooth. Since $u(t) \in C([0, T]; H)$, it is clear from the definition that the functions $t \to (B(u(t), e_i), e_j)$ and $t \to (B(e_i, e_j), u(t))$ are continuous in $t$, that is, (4.5) holds. Thus, all the requirements of Condition 4.1 are fulfilled.

**Lemma 8.2.** Let $u_0 \in H$ and $u(t) = S_t u_0$, where $S_t$ is an operator of the semigroup corresponding to (8.5). Then for any $v_0 \in H$ (8.13) has a unique solution such that for $0 \leq t \leq T$

$$\|v(t)\|_1 \leq C \|v_0\|_1, \quad \int_0^T \|v(t)\|_1^2 \, dt \leq C \|v_0\|_1^2, \quad C = C(||u_0||, T). \tag{8.19}$$

The shift operator $G_t : v_0 \to v(t)$ corresponding to (8.13) is the differential $S_t(u_0)$ of $S_t$ at the point $u_0$. The operator $S_t : H \to H$ is uniformly differentiable on any bounded set $B$ in $H$. The operator $S_t(u)$ satisfies the Hölder condition (6.3) on $B$. 

}\n
Proof. The function $u(t)$ satisfies (8.10) for any $T > 0$. Multiplying (8.13) by $v(t)$, using Gronwall's inequality, and applying (8.10) we obtain (8.19). From these estimates (which are also valid for the Galerkin approximations to (8.13)) it is easy to deduce by standard means the existence and uniqueness of the solution of (8.13) in the given class of functions. Hence, $G_t$ is well defined. To prove that $G_t = S'_t(u_0)$ we follow the scheme outlined in §7.

The equation (7.7) corresponding to (8.5) takes the form
\[
(8.20) \quad \partial_t w = v \Pi \Delta w - B(w, u + v) - B(u+v, w) - B(w, w) - B(v, v).
\]

Multiplying this equation by $w$ and using (8.15), we find that
\[
(8.21) \quad \frac{1}{2} \partial_t \|w\|^2 + \nu \|w\|^2 = -(B(w, u + v), w) - (B(v, v), w) = (-B(w, u + v), w) + (B(v, v), w).
\]

It follows from (8.21) and (8.17) that
\[
\partial_t \|w\|^2 + 2\nu \|w\|^2 \leq 2C (\|u + v\|_4 \|w\|_4 \|w\|_4 + \|w\|_4 \|v\|_4 \|v\|_4).
\]

Using (7.15) we deduce that
\[
(8.22) \quad \partial_t \|w\|^2 \leq C_2 \|u + v\|^2_4 \|w\|^2 + C_3 \|v\|^2_2 \|v\|^2_2, \quad \|w(0)\| = 0.
\]

Integrating this inequality with respect to $t$ and taking (8.19) into account we obtain (7.9), where $\gamma = 2$ and $\varphi(\tau) = C_2 \|u + v\|^2$. (7.10) follows from (8.10) and (8.19), and it is obvious that $M$ depends only on $\|u_0\|$ and $T$.

Using Proposition 7.1 we now obtain (7.11), from which it follows that $\bar{u} \in C$ and also that $S_t$ is uniformly differentiable on any bounded set in $H$.

To verify (6.3) we consider (7.12) corresponding to the equation (8.3). As in the derivation of (8.22) from (8.20) we obtain for the function $w = v_1 - v_2$ (the solution of (7.12)) the estimate
\[
\partial_t \|w\|^2 \leq C_3 \|u_1\|^2 \|w\|^2 + C_4 \|v_2\| \|v_2\|_1 \|u_1 - u_2\|_1 \|u_1 - u_2\|_1.
\]

Integrating this with respect to $t$ and using the continuous dependence of $u_i(t)$ on $u_i(0)$ in $H$ and also (8.19) we arrive at (7.9), where $\gamma = -1$, $\|E\| = \|E\|$, $C_0 = C_0 \|v_2(0)\|^2$, and $\|v_0\|$ is replaced by $\|u_2(0) - u_2(0)\|$.

By Proposition 7.1, we obtain (7.11) ($v_0 = u_i(0) - u_2(0)$), which yields (6.3).

We are now ready to estimate the dimension of the attractor.

**Theorem 8.2.** Let $\mathcal{A}$ be the maximal attractor of the Navier-Stokes system for zero boundary conditions. Then the Hausdorff dimension of $\mathcal{A}$ in $H$ is finite.

\[
(8.23) \quad \dim \mathcal{A} \leq C v^{-4}, \quad \forall v > 0, \quad v \leq v_0 \quad (C = C(\|/\|_1)).
\]

*Proof.* The existence of $\mathcal{A}$ and the fact that it is bounded in $H$ have been established in Theorem 8.1. To estimate its dimension we use Theorem 5.2, where $X = \mathcal{A}$ and $\{S_t\}$ is the semigroup corresponding to (8.5). By Lemma 8.2, the operators $S_t$ are uniformly differentiable on $\mathcal{A}$, and the differential
$S_t(u_0), u_0 \in \mathcal{H}$, is equal to the shift operator $G_t$ of (8.13). Since $\mathcal{H}$ is bounded in $H_1 = V$, the function $S_t u_0 = u(t)$ is bounded in $H_1$, uniformly in $t$, for any $u_0 \in \mathcal{H}$. Therefore, the conditions of Lemma 8.1 are satisfied and the operator $L(t) = A'(u(t))$ satisfies Condition 4.1. By (8.18), the estimate (4.11) has the form

\begin{equation}
(L(t) \nu, \nu) \leq -\frac{\nu}{2} (L_0 \nu, \nu) + h(t) \||v||^2,
\end{equation}

where $h(t) = C_1 v^{-1} ||u(t)||^2_1$. In this section the role of $Q$ in (4.1) is played by the operator $L_0 = -\Pi \Delta$. Hence, by (8.12),

\begin{equation}
\int_0^t h(\tau) d\tau \leq C_1 v^{-1} (t v^{-2} ||f||^2_{-1} + v^{-3} \gamma_1^{-1} ||f||^2_{-1}) = h^0_1.
\end{equation}

Thus, the second estimate in (5.6) holds, where $h^0_1$ is defined by (8.25). For $v^0_1$ in (5.6) we can take $v t / 2$ since $v(t) = v / 2$ in (4.11) by (8.24). As is well known, the eigenvalues $\gamma_j$ of $L_0 = Q$ in a bounded domain $\Omega \subset \mathbb{R}^2$ satisfy $\gamma_j > C_j, j > 0$. Therefore, in (5.7) we can take $\eta(N) = C_5 N$ for $\eta(N)$. Then $\gamma^{-1}(r) = C_5^{-1} r$. Hence all the hypotheses of Theorem 5.2 are satisfied, and when we substitute in (5.8) the values of $v^0_1$ and $h^0_1$ above, we obtain

\begin{equation}
\text{dim } \mathcal{H} \leq C_5^{-1} \left[ \frac{h^0_1}{v^0_1} \right] = C_5^{-1} \left[ \frac{C_4 v^{-3} \|f\|^2_{-1} + C_4 v^{-4} \gamma_1^{-1} \|f\|^2_{-1}}{v t / 2} \right]
\end{equation}

for any $t > 0$. Taking the limit $t \to +\infty$, we deduce (8.23).

2. We now consider the system (8.1) with the periodic boundary conditions (8.3). The spaces $H$ and $V$ now consist of the functions $u(x)$ defined on the torus $T^2(\alpha_0) = T^2$ and satisfying $\text{div } u(x) = 0$ as well as (8.4); the operator $\Pi$ is now an orthogonal projection in $(L_2(T^2))^2$. As in §8.1, we rewrite (8.1) in the form (8.5). In the periodic case (8.7) also defines a semigroup \{$S_t$\}, $S_t : H \to H$. All the assertions 1)-6) of Theorem 8.1 hold. In particular, there is a maximal attractor $\mathcal{A}$. For its dimension we can now derive an estimate stronger than (8.23) for zero boundary conditions.

**Theorem 8.3.** Let $\mathcal{A}$ be the maximal attractor of (8.1) under periodic boundary conditions, $g \in H_1(T^2(\alpha_0))$. Then the Hausdorff dimension of $\mathcal{A}$ for $\nu \leq K_0$ and $\alpha_0 \leq K_0$ is subject to the estimate

\begin{equation}
\text{dim } \mathcal{A} \leq C_0 \alpha_0^{-1} v^{-2} \left[ \|g\| (\log(v^{-1} \|g\|^{1/2} + 2))^{1/2} + \|g\|^{-1} \right].
\end{equation}

The basic role in the derivation of (8.26) is the fact that in the periodic case we can obtain a better estimate of the solutions on the attractor than (8.11) and (8.12).

**Lemma 8.3.** Let $\mathcal{A}$ be the maximal attractor for the semigroup \{$S_t$\} corresponding to (8.1) with periodic boundary conditions. Then there is a constant $C$ independent of $v$ and $\alpha_0$ such that for any $u \in \mathcal{A}$

\begin{equation}
\|u\|^2_1 \leq C \|g\|^2 \gamma_1^{-1} v^{-2},
\end{equation}
Attractors of partial differential evolution equations and estimates of their dimension

and

\[(8.28) \quad \int_0^T \| u(t) \|^2 dt \leq C T \nu^{-2} \| u \|^2 + C \nu^{-3} \gamma_1^{-1} \| g \|^2,\]

where \( u(t) = S_t u \). Here we may assume without loss of generality that \( \text{div} \ g = 0 \), that is, \( \Pi g = g \).

**Proof.** We introduce the two-dimensional curl operator \( Rv = \partial_1 v_2 - \partial_2 v_1 \), \( v = (v_1, v_2) \). Applying \( R \) to (8.1) and setting \( Ru = z, Rg = \varphi \), we obtain

\[ \partial_t z - \nu \Delta z + u_1 \partial_1 z + u_2 \partial_2 z = \varphi. \]

Multiplying this equation by \( z \) and integrating over \( \Omega \), we obtain by analogy with (8.9) and (8.10):

\[(8.29) \quad \| z(T) \|^2 + \nu \int_0^T \| z(t) \|^2 dt \leq T \nu^{-1} \| \varphi \|^2 + \| z(0) \|^2 \quad (T^+ > 0 \text{ by } (8.4)).\]

Expanding \( u \in H_1 \) in a Fourier series we easily deduce that

\[(8.30) \quad \| u(t) \|^2 \leq C \| Ru(t) \|^2 = C \| z(t) \|^2, \quad \| u \|^2 \leq C \| z \|^2 \quad (\text{div} \ u = 0).\]

Since \( S_t \mathcal{A} = \mathcal{A} \), for an arbitrarily large \( t \) we can find a \( u_0 \) such that \( S_t u_0 = u \). Setting \( z(t) = Ru \) and \( z(0) = Ru_0 \) in (8.29), we obtain, by (8.31),

\[(8.32) \quad \| u(t) \|^2 \leq C \| Ru(t) \|^2 = C \| z(t) \|^2 \leq \nu^{-1} \| \varphi \|^2 + \| z(0) \|^2 \quad (t \to +\infty).\]

Now \( \| Ru_0 \|^2 \leq C \| u_0 \|^2 \leq M_1 \) for any \( u_0 \in \mathcal{A} \). Since \( t \) in (8.32) can be taken arbitrarily large, as \( t \to +\infty \) we deduce (8.27). Here we must use the fact that \( \| \varphi \|^2 \leq \Pi g \leq \| g \|^2 \). Estimating the term \( \| z(0) \|^2 \) in (8.30) as follows:

\[(8.33) \quad \| z(0) \|^2 = \| Ru(0) \|^2 \leq \| u(0) \|^2 \leq C \| g \|^2 \gamma_1^{-1} \nu^{-2},\]

(in the last inequality we have used (8.27)), we obtain the required estimate (8.28) from (8.30) and (8.33).

The proof of Theorem 8.3 is similar to that of Theorem 8.2. All the hypotheses of Theorem 5.2 can be verified (as before, \( Q = L_0 \)). However, since the base of the rectangle \( \Omega_0 \) is of length \( 2\pi/\alpha_0 \), where \( \alpha_0 \) is a small parameter, some estimates will have to be altered. For example, for the eigenvalues of \( L_0 \) in \( \Omega_0 \) we have \( \gamma_j \geq C \alpha_0 j \), therefore, in (5.7) we take \( \eta(N) = C_\alpha N \). To choose the appropriate \( h(t) \) in (4.11), we have to estimate the quadratic form

\[(8.34) \quad (L(t)v, \upsilon) = -\nu (L_0 \upsilon, \upsilon) + (B(v, u(t)), \upsilon).\]

To estimate the last term in (8.34) we prove the following lemma:
Lemma 8.4. Let \( u \in H_2 \) and \( v \in H_1 \). Then for any \( R \geq 2 \) and \( \alpha_0 < 1 \),
\[
(8.35) \quad |(B(v,u),v)| \leq C_1 \left( \sqrt{\log R} \| u \|_2 \| v \|_2 + R^{-1} \| u \|_2 \| v \|_4 + \| u \|_4 \| v \|_4 \right),
\]
where \( C_1 \) is independent of \( R \) and of \( \alpha_0 < 1 \).

Proof. To simplify the notation we write \( \alpha \) instead of \( \alpha_0 \). We expand the functions in Fourier series on \( T^2 = T_2^\alpha \). The Fourier coefficient \( \hat{f}(\xi) \), \( \xi \in \mathbb{Z}^2 \), of a function \( f(x) \) is defined by
\[
\hat{f}(\xi) = \frac{\alpha}{4\pi^2} \int_0^{2\pi} f(x) e^{-ix_1x_1 -ix_2x_2} \, dx_1 dx_2 \quad (i = \sqrt{-1}).
\]
It is obvious that \( \partial_{\xi_1} \hat{\xi}(\xi) = i\alpha_1 \hat{x}(\xi), \partial_{\xi_2} \hat{\xi}(\xi) = i\xi_2 \hat{\xi}(\xi) \), and \( \Delta \hat{\xi}(\xi) = -|\xi|_2^2 \hat{\xi}(\xi) \), where \( |\xi|_2 = \alpha_1^2 + \alpha_2^2 \). The Parseval equality
\[
(f_1, f_2) = \int_{T^2} f \overline{f_2} \, dx = \frac{4\pi^2}{\alpha} \sum_\xi \hat{f}_1(\xi) \overline{\hat{f}_2(\xi)}.
\]
holds. Using this we obtain
\[
|(B(v,u),v)| = \frac{4\pi^2}{\alpha} \sum_{i,j=1}^2 \langle \overline{v} \hat{v}_j(\xi) \overline{\hat{u}_j(\xi)} \rangle.
\]
Obviously, for \( R \geq 2 \),
\[
(8.36) \quad \sum_\xi |\overline{v}_j(\xi)| \langle \overline{\hat{u}_j(\xi)} \rangle \leq \sup_{\xi} |\overline{v}_j(\xi)| \sum_{0 < |\xi|_2 < 1} |\hat{u}_j(\xi)| + \sum_{0 < |\xi|_2 < R} |\hat{u}_j(\xi)| + \sup_{|\xi|_2 \geq R} |\overline{v}_j(\xi)| \sum_{|\xi|_2 \geq R} |\hat{u}_j(\xi)|.
\]
We observe that we have \( \xi_2 = 0 \) and \( |\xi|_1 = \alpha_1 |\xi_1| \) for \( |\xi|_2 < 1 \). Therefore,
\[
(8.36') \quad \sum_{0 < |\xi|_2 < 1} |\hat{u}_j(\xi)| \leq \left( \sum_{0 < |\xi|_2 < 1} |\overline{v}_j(\xi)| \right)^{1/2} \left( \sum_{0 < |\xi|_2 < 1} |\hat{u}_j(\xi)|^2 \right)^{1/2} \leq \sqrt{\frac{2}{\alpha}} \| u \|_1 \| \overline{v}(\xi) \|_2 = \sqrt{\frac{2}{(2\pi)^2}} \| u \|_1.
\]
It is easy to see that
\[
(8.36') \quad \sum_{0 < |\xi|_2 < R} |\hat{u}_j(\xi)| \leq C \sqrt{\log R} \| u \|_2^2.
\]

The Fourier transform of a product of functions \( f_1(x)f_2(x) \) is given by
\[
(8.37) \quad \hat{f}_1\hat{f}_2(\xi) = \sum_\eta \hat{f}_1(\xi - \eta) \hat{f}_2(\eta).
\]
Using the Cauchy-Bunyakovsky inequality and the Parseval equality, we obtain
\[
\sup_\xi |\hat{f}_1\hat{f}_2(\xi)| \leq \frac{C}{4\pi^2} \| f_1 \| \cdot \| f_2 \| \| \cdot \| = \| L_2(T^2(\alpha)).
\]
It follows from this estimate that
\begin{equation}
(8.38) \quad \sup_{\xi} |\tilde{v}(\xi)| \leq \frac{\alpha}{(2\pi)^3} ||v||^2.
\end{equation}

Using the inequality
\[ ||v||^2 \leq ||v||^2 + ||v||^2 + C ||u||^2 \]
and (8.37), we obtain by analogy with (8.38)
\begin{equation}
(8.39) \quad \sup_{\xi} |\tilde{v}(\xi)| \leq \frac{\alpha}{(2\pi)^3} ||v||^2.
\end{equation}

The estimate (8.35) follows immediately from (8.36), (8.36'), (8.36''), (8.38), and (8.39).

Remark 8.1. The arguments used in the proof of (8.35) are similar to those in [44] (see [43]).

We carry on with the proof of Theorem 8.3. From (8.34) and (8.35) it follows that
\begin{equation}
(L(t)v,v) \leq -v ||v||^2 + C_4 \sqrt{\log R} ||u||^2 ||v||^2 + C_4 R^{-1} ||v||^2 + C_4 ||u||^2 ||v||^2 \leq
\end{equation}
\[ \leq -\frac{v}{2} ||v||^2 + \left( C_4 \sqrt{\log R} ||u||^2 + C_4 R^{-1} ||v||^2 + C_4 ||u||^2 ||v||^2 \right) \]
\[ + C_4 R^{-2} ||v||^2 \int_0^t \left( ||u(\tau)||^2 + C_4 \sqrt{\log R} ||u||^2 + C_4 R^{-1} ||v||^2 \right) d\tau \)
\[ + C_4 R^{-2} ||v||^2 \int_0^t ||u(\tau)||^2 d\tau + C_4 \sqrt{\log R} \left( \int_0^t ||u(\tau)||^2 d\tau \right)^{1/2} \]

Using (8.28) and (8.12) to estimate the integrals with respect to \( \tau \), we find that (5.6) is satisfied, where
\[ h^2(t) = t \left[ C_4 \sqrt{\log R} ||v||^2 + C_4 R^{-1} ||g||^2 + C_4 ||v||^2 + C_4 ||u||^2 + \varphi(t) \right], \]
with \( \varphi(t) \to 0 \) as \( t \to +\infty \). For \( v^2(t) \) in (5.6) we can take \( v(t/2) \). Substituting these expressions in (5.8) where \( \eta(N) = C_2 \alpha_0 N \), we obtain
\begin{equation}
\dim \mathcal{V} \leq \lim_{t \to +\infty} C_2^{-1} \alpha^{-1} (h^2(t)) \leq
\end{equation}
\[ \leq C_2 \alpha_0^{-1} [\sqrt{\log R} ||v||^2 + R^{-1} ||g||^2 + \alpha^{-1} ||g||^2]. \]

Taking \( R = v^{-1} ||g||^{1/2} + 2 \) we now arrive at an estimate for the dimension of the attractor
\begin{equation}
\dim \mathcal{V} \leq C_2 \alpha_0^{-1} v^{-2} [||g|| (1 + \sqrt{\log v^{-1}} ||g||^{1/2} + 2) + ||g||],
\end{equation}
which completes the proof of Theorem 8.3.
In particular, if \( g \) is a function independent of \( x_1 \), then \( g(x_2) \) is defined on \( T^2(\alpha_0) \) for all \( \alpha_0 \). Here \( \| g \|_{-1} = C_4 \alpha_0^{1/2} \) and \( \| g \| = C_5 \alpha_0^{1/2} \). Thus, for \( g(x) = g(x_2) \)

\[
\dim \mathcal{Y} \leq C_6 \alpha_0^{-3/2} v^{-2} \sqrt{\ln (v^{-1} \alpha_0^{-1/2}) + 2}.
\]

**Remark 8.2.** Let \( \Pi_N \) be the projection operator in \( H \) onto the subspace with the basis consisting of the first \( N \) eigenfunctions of \( \Pi \Delta \). Let \( u(t) \) be the solution of the two-dimensional Navier-Stokes system and \( \Pi_N u(t) \) its projection. In [45] and [43] the authors study the problem for what \( N \) the asymptotic behaviour of \( \Pi_N u(t) \) as \( t \to +\infty \) determines uniquely that of \( u(t) \) as \( t \to +\infty \). Estimates of \( N \) in terms of \( \nu \) are obtained in [43]. They are of the same order of magnitude as (8.23) and (8.26), which indicates that there is a close connection between the problems of estimating \( N \) and the dimension of the attractor.

3. We now come to a lower bound for the Hausdorff dimension of the attractor \( \mathcal{Y} \) of the system (8.1) under the periodic boundary conditions (8.3). For this purpose we use an example constructed by Yudovich [18] in which the instability of parallel flows of a viscous incompressible fluid under space-periodic perturbations is established. Let \( g(x) = (-\gamma g_1(x_2), 0) \) in (8.1), where \( g_1(x_2) \) is a \( 2\pi \)-periodic function whose average value over the period is zero, that is, \( g(x) \) satisfies (8.4). The problem (8.1), (8.3) has a unique stationary solution \( u = z_0, \rho = p_0 (p_0 = \text{const}) \)

\[
(8.41) \quad z_0 (x) = (u_1 (x) u_2 (x)) = (\gamma v^{-1} U (x_2), 0),
\]

\[
\begin{align*}
\partial_2^2 U (x_2) &= g_1 (x_2), \\
\int_{-\pi}^{\pi} U (x_2) \, dx_2 &= 0,
\end{align*}
\]

\[
\int_{-\pi}^{\pi} g_1 (x_2) \, dx_2 = 0, \quad U (x_2 + 2\pi) = U (x_2).
\]

Below we estimate the dimension of the subspace \( E_+ \) of unstable directions at a stationary point \( z_0 \). For this purpose we study the variational equation of the system (8.1) at \( z_0 \), which for the current function \( \varphi(x, t) = -v_2, \partial_2 \varphi = v_1 \) has the following form (see [18]):

\[
(8.42) \quad \partial_t \Delta \varphi + \gamma v^{-1} U(x_2) \Delta \varphi - U^* (x_2) \partial_1 \varphi - \nu \Delta^2 \varphi = 0.
\]

We look for solutions of this equation in the form

\[
(8.43) \quad \psi (x, t) = e^{\psi_{0} t + i \alpha x^1} \psi_2 (x_2), \quad \psi (x_2 + 2\pi) = \psi (x_2),
\]

where \( \alpha = k \alpha_0, \quad k \in \mathbb{N} \). Substituting this expression for \( \psi \) in (8.42) we obtain the Orr-Sommerfeld equation

\[
(8.44) \quad \sigma \left( \frac{d^2}{dx_2^2} - \alpha^2 \right) \psi + i \alpha \lambda \left[ U \left( \frac{d^2}{dx_2^2} - \alpha^2 \right) \psi - U^* \psi \right] - \left( \frac{d^2}{dx_2^2} - \alpha^2 \right)^2 \psi = 0,
\]

where \( \lambda = \gamma v^{-2} \). To the values of \( \sigma \) with \( \text{Re} \sigma > 0 \) and the appropriate solutions \( \psi (x_2) \neq 0 \) of (8.44) there correspond unstable solutions \( \varphi(x, t) \) of
the form (8.43) (that is, \( \varphi \to \infty \) as \( t \to +\infty \)) of (8.42) and to the current functions \( \varphi \) there correspond unstable solutions \( v(x, t) = (v_1, v_2) \) of the variational equation of the system (8.1). These \( v(x, t) \) are eigenfunctions of the operator \( S_t(x_0) \) corresponding to eigenvalues \( e^{\nu t} \), \( |e^{\nu t}| > 1 \). Therefore, a lower bound for the number of unstable solutions \( \psi \) of (8.44) gives us a lower bound for the dimension of \( E_+ \) (see Theorem 6.1 and Corollary 6.2).

We take \( \lambda = \gamma v^-2 \), that is, choose a fixed \( v \). As was shown in [18], for sufficiently small \( \alpha : |\alpha| = |k\alpha_0| < \delta \), there are corresponding values \( \sigma = \sigma(\alpha) \) and \( \psi = \psi(x_2; \alpha) \) satisfying (8.44), which can be expanded in convergent power series in \( \alpha \):

\[
\sigma = \sum_{l=0}^{\infty} \alpha^l \sigma_l, \quad \psi = \sum_{l=0}^{\infty} \alpha^l \psi_l(x_2).
\]

Here \( \sigma_0 = 0, \sigma_1 = 0, \sigma_2 = \lambda^2/2\pi \int_{-\pi}^{\pi} (\theta'(x_2))^2 dx_2 - 1 \), where \( \theta(x_2) \) is a periodic solution of the equation \( \partial \tilde{\theta}(x_2) = U(x_2) \) with zero average over the period.

Thus, \( \sigma_2 > 0 \) for sufficiently large (but fixed) \( \lambda \), consequently, \( \text{Re} \sigma > 0 \) for small \( \alpha \) and \( \text{Re} \sigma \) increases with \( \alpha \) for small \( \alpha \). To each value \( k \in \mathbb{N} \) such that \( \alpha_0 k < \delta \), there corresponds a \( \sigma = \sigma(\alpha_0 k) \) and a solution of (8.44) \( \psi = \psi(x; \alpha_0 k) \) such that the corresponding \( \sigma \) are distinct. The number of such solutions \( (\sigma, \psi) \) is certainly not less than that of integers \( k \) with \( k\alpha_0 < \delta \) or \( k < \delta/\alpha_0 \). Hence, \( \text{dim} E_+ \geq |\delta/\alpha_0| \). Since the semigroup \( \{S_t\} \) corresponding to (8.1), (8.3) has a Fréchet differential in \( H \) according to Lemma 8.2 (applied to the periodic case) and its differential \( S'_t(u) \) satisfies the Hölder condition (6.3), the hypotheses of Theorem 6.1 are

\[
\dim \mathcal{A} \geq \dim E_+ \geq |\delta/\alpha_0|.
\]

**Remark 8.1.** We consider the case when \( f \) in (8.5) depends periodically on \( t \) with a period \( I \) and \( f(t) \) is a continuous function of \( t \) in \( H_1 \). In this case there is defined a discrete semigroup \( \{S_{tk}\} \) of shift operators by time intervals that are multiples of \( I \). All the arguments in this case are parallel to the case of a continuous semigroup. An attractor \( \mathcal{A} \) exists in the periodic case, which is invariant under \( S_{tk} \), and the condition of attraction in Definition 1.1 holds for \( t = tk \) (\( k = 0, 1, ... \)).

We set \( \mathcal{A} = \bigcup_{t \in [0, I]} \mathcal{S}_t \mathcal{A}_p \). It is not hard to observe that \( \mathcal{A} \) is invariant under all \( S_t \), \( t \geq 0 \), corresponding to (8.5) with periodic right-hand side (the family \( \{S_t\} \) in this case does not form a semigroup). It is easy to see that \( \mathcal{A} \) has the property of attraction in Definition 1.1 for any \( t, t \to +\infty \) and not only for \( t = tk \). (This follows from the fact that \( S_t \) is uniformly continuous in \( t \) for \( t \in [0, I] \).) The upper bounds for \( \dim \mathcal{A} \) obtained in Theorems 8.2 and 8.3 are valid. The only difference is that the estimates (8.12) and (8.28) now contain instead of \( \|f\|_{-1} \) and \( \|f\| \) the quantities \( \max_t \|f(t)\|_{-1} \) and \( \max_t \|f(t)\| \).
§9. Upper and lower bounds for the Hausdorff dimension of attractors of parabolic equations and parabolic systems

This section consists of three parts. In §9.1 we consider the parabolic equation in Example 2.2 and the systems of the type of chemical kinetics in Example 2.3, and we derive upper bounds for the Hausdorff dimension of the attractor $\mathfrak{M}$, which depend on parameters like the viscosity $\nu$ and the spectral parameter $\lambda$ of the equations. The resulting estimates have the form $\dim \mathfrak{M} \leq C \nu^{-n/2} \lambda^{n/2}$, that is, their rate of growth as $\nu \to 0$ and $\lambda \to +\infty$ is the same as for linear parabolic equations. These estimates are based on the results of §5. In §9.2 we use the results of §6 to derive a lower bound for the dimension of the attractor for the system of equations of the type of chemical kinetics: $\dim \mathfrak{M} \geq c \nu^{-n/2} \lambda^{n/2}$ ($\lambda$ is fixed). In §9.3 we obtain a lower bound for the dimension of the attractor of the parabolic equation in Example 2.2: $\dim \mathfrak{M} \geq c \nu^{n/2}$ ($\nu$ is fixed).

1. We consider a parabolic equation

\begin{equation}
\frac{\partial u}{\partial t} = \nu \sum_{i,j=1}^{n} \partial_{i} \left( a_{ij} (x) \partial_{j} u \right) + \sum_{i=1}^{n} \partial_{i} \left( b_{i} (x, u) \right) - f(x, u) + \lambda u,
\end{equation}

\begin{equation}
|b_{i} (x, u)| \leq b_{0}, \quad x \in \Omega, \quad u \in \mathbb{R}.
\end{equation}

We assume that all the conditions of Example 2.2 are satisfied and also that

\begin{equation}
\|b_{i} (x, u)| \leq b_{0}, \quad x \in \Omega, \quad u \in \mathbb{R}.
\end{equation}

It is obvious that (9.1) is a special case of (2.11). We assume that the function $a_{0}(x, u) = f(x, u) - \lambda u + \sum \partial b_{i}/\partial x_{i}$ satisfies (2.14) for any $\lambda > 0$.

Then by Theorem 2.6, the semigroup $\{S_{t}\}$, $S_{t} : E \to E$ (see §2) corresponding to (9.1) has a maximal attractor $\mathfrak{M}$. Below we give an estimate for the Hausdorff dimension of $\mathfrak{M}$ depending on $\lambda$ and $\nu$, as $\lambda \to +\infty$ and $\nu \to 0$.

The operator $L(t) \equiv A'(u(t))$ in the variational equation (7.16) corresponding to (9.1) has the form

\begin{equation}
L(t) v = \nu \sum_{i,j=1}^{n} \partial_{i} \left( a_{ij} \partial_{j} v \right) + \sum_{i=1}^{n} \partial_{i} \left( b_{i}' (x, u (t) ) \right) v - f'(x, u (t) ) v + \lambda v.
\end{equation}

Proposition 9.1. Let

\begin{equation}
f'(x, u) \geq - f^{0}.
\end{equation}

Then

\begin{equation}
(L(t) v, v) \leq - \frac{\nu}{2} \| \nabla v \|^2 + \frac{b_{0} C}{\nu} \| v \|^2 + \lambda \| f^{0} \| v \|^2.
\end{equation}

This inequality is proved by multiplying (9.3) by $v$ and integrating by parts, and applying (2.12), (9.2), and (9.4).

Theorem 9.1. Let $\mathfrak{M}$ be the maximal attractor of $\{S_{t}\}$ corresponding to (9.1) and assume that (9.2) and (9.4) hold. Then the Hausdorff dimension of $\mathfrak{M}$
in $L_2(\Omega)$ is bounded by
\[
\dim \mathcal{Y} \leq C_3(b^0)^n/2v^{-n} + C_4\lambda^{n/2}v^{-n/2}, \quad \lambda > 1, \quad v > 0.
\]

Proof. We use Theorem 5.2. By Theorem 7.1, $S_t$ is quasidifferentiable on $\mathcal{Y}$ and its quasidifferential $S'_t(u_0)$ is equal to the linear shift operator corresponding to (4.2), where $L(t)$ is defined by (9.3). It follows from (9.5) that (4.11) holds with $\nu(t) = \nu\mu_0/2$, $h(t) = b^0Cv^{-1} + \lambda + f^0$, and $Q = -\Delta$, where $\Delta$ is the Laplace operator with zero boundary conditions. It follows from the known asymptotic behaviour of the eigenvalues $\gamma_i$ of the Laplace operator that
\[
\frac{1}{N} \sum_{j=1}^{N} \gamma_j \geq C_0 N^{2/n}.
\]

Therefore, in (5.7) we can choose $\eta(N) = C_0 N^{2/n}$. It is obvious that $\eta^{-1}(r) = C_1 r^{n/2}$. According to (5.8) we find for $\dim \mathcal{Y}$ as $t \to +\infty$, that
\[
\dim \mathcal{Y} \leq C_4 \left( \frac{b^0Cv^{-1} + \lambda + f^0}{\nu\mu_0/2} \right)^{n/2} \leq \left( \frac{b^0C_2}{v^2} + \frac{(\nu + \lambda)C_2}{v} \right)^{n/2},
\]
which yields (9.6).

We note that systems including only one of the parameters $\lambda$ or $\nu$ are frequently encountered. In this case we simply assume that the second parameter in (9.6) is fixed.

Remark 9.1. If (9.1) contains no first-order terms, that is, $b^0 = 0$, then the rate of growth of $\dim \mathcal{Y}$ as $\lambda \to +\infty$ and $\nu \to 0$ is the same as for linear parabolic equations.

Remark 9.2. If we consider the equations (2.11) without the additional conditions (9.2) and the requirement that the principal part is linear, then the dimension of $\mathcal{Y}$ remains finite, but the estimates of the Hausdorff dimension of $\mathcal{Y}$ depending on $\lambda$ and $\nu$ take a more complicated form, since the estimates of $(L(t)u, u)$ include estimates of the norms of $u$ and $\nabla u$ on $\mathcal{Y}$, which in turn depend on $\lambda$ and $\nu$.

We now consider the system of equations of the type of chemical kinetics (2.21) in Example 2.3, where $a = \nu a_0$, $a_0$ is a matrix for which $(a_0 + a_0') \geq \mu_0 I$ and $\nu > 0$. Furthermore, we assume that (9.2) and (9.4) hold, where $b'_{tu}$ and $f'_u$ are matrices. Repeating verbatim the proof for the scalar case and using Theorem 7.2 instead of Theorem 7.1, we arrive at the following conclusion:

**Theorem 9.2.** Under the conditions specified above and the hypotheses of Theorem 7.2, the Hausdorff dimension in $(L_2(\Omega))^m$ of the attractor $\mathcal{Y}$ for $\{S_t\}$ corresponding to (2.21) satisfies (9.6).
Remark 9.3. The boundary condition \( u|_{\partial \Omega} = 0 \) in (2.21) can be replaced by

\[
\frac{\partial u}{\partial l} |_{\partial \Omega} = 0, \quad l \text{ is the normal to } \partial \Omega.
\]

Here Theorem 9.2 and the estimate (9.6) remain valid.

2. We now obtain a lower bound for the Hausdorff dimension of the attractor for the system of the type of chemical kinetics (see [33]). For this purpose we use the results of §6. We consider the system (2.21) for the case when \( f(x, u) \) is independent of \( x \) and \( b_i = 0 \):

\[
\frac{\partial u}{\partial t} = \nu \Delta u - f(u) + \lambda u - g \equiv A(u),
\]

where \( g \) is independent of \( x \), subject to the boundary condition (9.8). To use the results of §6 we have to find a stationary solution of (9.9), that is, a solution \( z \) of the equation \( A(u) = 0 \) satisfying (9.8). We look for such a solution in the form of a constant. Obviously, such a \( z \) is the solution of the equation \( f(z) - \lambda z + g = 0 \). It follows from (2.23) that such a \( z \) always exists. We fix one of them.

We consider the variational equation corresponding to (9.9):

\[
\frac{\partial v}{\partial t} = \nu \Delta v - f'(z)v + \lambda v \equiv A'(z)v
\]

with the boundary condition (9.8). According to Theorem 6.2, to obtain a lower bound for \( \text{dim} \mathfrak{A} \) we need one for \( \text{dim} E^+, \) where \( E^+ \) is the invariant subspace of the operator \( A'(z) \) corresponding to the eigenvalues \( \xi_i \) of \( A'(z) \) with \( \Re \xi_i > 0 \).

**Theorem 9.3.** Let \( z \in \mathbb{R}^m \) be a solution of the equation \( -f(z) + \lambda z = g \) and suppose that the matrix \( \Phi = -f'(z) + \lambda I \) has at least one eigenvalue \( \mu^0 = \xi + i\mu \) with \( \xi > 0 \).

Then for small \( \nu > 0 \)

\[
\text{dim} E^+_0 \geq c\nu^{-n/2}, \quad c > 0.
\]

**Proof.** Let \( \{\psi_i(x)\} \) be an orthonormal basis of eigenfunctions of the scalar Laplace operator

\[
\Delta \psi_i = -\omega_i \psi_i, \quad 0 = \omega_1 < \omega_2 \leq \omega_3 \leq \ldots,
\]

with the boundary condition (9.8). Following [33], we look for the eigenvectors \( w_i \) of \( A'(z) \) defined by (9.10), that is, for the solutions of

\[
A'(z)w_i = \xi w_i
\]

in the form \( w_i(x) = \psi_i(x)\overline{p}_i, \overline{p}_i \in \mathbb{C}^m \). Substituting this expression in (9.13) and bearing (9.12) in mind we obtain

\[
(\nu \omega_i a_0 + \Phi) \overline{p}_i = \xi \overline{p}_i.
\]

There is a non-zero \( \overline{p}_i \) if \( \xi \) is a root of the equation

\[
\det (\nu \omega_i a_0 + \Phi - \xi I) = 0.
\]
It follows from the hypotheses of the theorem that (9.15) for \( \nu = 0 \) has at least one root \( \xi \) with \( \text{Re} \xi = \xi > 0 \). Therefore, we can find a \( \delta > 0 \) such that (9.15) for \( \nu \omega_i < \delta \) has a root \( \xi_i = \xi_i(\nu) \), \( \text{Re} \xi_i > 0 \). To each such root there corresponds a vector \( \vec{p}_i \), a solution of (9.14), and an eigenfunction \( w_i = \psi_i \vec{p}_i \), a solution of (9.13) with the eigenvalue \( \xi_i \). We count the number of \( i \) for which \( \nu \omega_i < \delta \). It follows from the standard asymptotic expansion \( \omega_i \sim c_i^{2/n} \) that this inequality is certainly satisfied for

\[
1 < i < C_1\delta^{n/2}\nu^{-n/2} = C_2\nu^{-n/2}.
\]

**Corollary 9.1.** Under the assumptions of Theorem 9.3, Example 2.3, and Theorem 7.3, the dimension of the attractor corresponding to (9.9) satisfies the inequality

\[
\dim \mathcal{A} \geq c\nu^{-n/2}, \quad c > 0.
\]

**Proof.** By Theorem 7.3 (which holds for the boundary conditions (9.8)), the operators \( S_t: (H^1(\Omega))^m \to (H^1(\Omega))^m \) of the semigroup \( \{S_t\} \) corresponding to (9.9) are differentiable, and their differentials satisfy (6.3). It is easy to show that the operator \( S'_t(z) \) is completely continuous in \( H_1 \). Therefore, bearing in mind Remark 6.2 we find that all the hypotheses of Theorem 6.1 are satisfied. The set \( \Lambda \) is bounded in \( H_y \), consequently also in \( H \). By Corollary 6.1 and Theorem 6.1, \( \dim \mathcal{A} \geq \dim E_+ \). And from the proof of Theorem 6.2 it is clear that \( \dim E_+ \geq \dim E_0 \). Making use of (9.11) we then obtain (9.17).

**Remark 9.4.** The equation \( \lambda z - f(z) = \Phi \) for large \( \lambda \) has a solution \( z = z(\lambda) \) of order \( \lambda^\eta \). (This is easy to establish by assuming \( \Phi = 0 \) and using .. form \( z_0 = z_0\lambda^{-1} + z_0(\lambda)\lambda^{-2} \) (see \$9.3).) Here \( \Phi = -f(z_0(\lambda)) + \lambda I \) has eigenvalues with \( \text{Re} \mu > 0 \) for large \( \lambda \). Since (9.15) can be rewritten in the form

\[
\det (-\nu \omega_1 \lambda^{-1} a_0 - \lambda^{-1} f'(z_0(\lambda)) + I - \xi_1 I),
\]

where \( \xi_1 = \xi_1 \lambda^{-1} \), it is easy to see that (9.13) certainly has solutions for \( \lambda \geq \lambda_0 \gg 1 \) and \( \nu \omega_1 \lambda^{-1} \ll \delta_1 \). Therefore, \( c \geq \lambda_0^{n/2} \) in (9.17) for large \( \lambda \) and small \( \nu \). Hence, the lower bound for \( \dim \mathcal{A} \) has the same form as (9.6) for \( b^0 = 0 \):

\[
C\lambda^{n/2}\nu^{-n/2} \geq \dim \mathcal{A} \geq c_0\lambda^{n/2}\nu^{-n/2}.
\]

**Remark 9.5.** The estimates (9.17), as is easy to see, hold even if the term \( \lambda u \) in (9.9) is replaced by \( \lambda C u \), where \( C \) is a matrix for which \( (C + C^*) > 0 \). The upper bound (9.18) also remains valid under zero boundary conditions when \( \lambda I \) is replaced by \( \lambda C \), as above. The lower bound for the dimension of the attractor subject to zero boundary conditions is discussed in Remark 9.9.

**Remark 9.6.** If the semigroup \( \{S_t\} \) corresponding to (9.9) takes the set \( K \) consisting of functions with non-negative components to itself and if the
vector $z$ in the conditions of Theorem 9.3 has positive components, then
\[
\dim \mathfrak{A}_K \geq cv^{-n/2} \quad (\text{see Remarks 1.1 and 2.2}).
\] This follows from the fact that the local unstable manifold $M(\Lambda, z, S_t, r)$ constructed in Theorem 6.1 is contained within $K$ if $z$ lies inside $K$.

3. We now consider a special case of (9.1):

\[
\begin{align*}
\tag{9.19} 
\partial_t u &= \Delta u + \lambda u - f(u) - g(x) = A(u), \quad x \in \Omega, \quad \lambda > 0, \\
\tag{9.19'} 
\text{subject to zero or periodic boundary conditions} \\
\tag{9.19''} 
\end{align*}
\]

We assume that $f(u)$ satisfies (2.14) and that $f \in C^3(\mathbb{R})$ and $f(0) = f'(0) = 0$. These equalities do not impose a restriction, since they can be always made to hold by an appropriate change of $\lambda$ and $g(x)$. Below we derive a lower estimate for the dimension of $\mathfrak{A}$ as $\lambda \to \infty$. As will be shown in §11, the principal term of this estimate (in $\lambda$) is identical with the upper bound of the dimension of the attractor.

**Theorem 9.4.** We assume that (9.19) for any $\lambda \geq 0$ in some unbounded set $Y \subset \mathbb{R}$ has a stationary solution $z = z(x, \lambda)$ for which

\[
\tag{9.20} ||z(\lambda)||_{C(\Omega)} \leq M \forall \lambda \in Y, \quad M \text{ is independent of } \lambda.
\]

Then for $\lambda \in Y$ the following estimate holds:

\[
\tag{9.21} \dim \mathfrak{A} \geq N(\lambda - C, -\Delta) \quad (\mathfrak{A} = \mathfrak{A}(\lambda)),
\]

where $N(\lambda, -\Delta) = N(\lambda)$ is the number of eigenvalues of $-\Delta$ (subject to the appropriate boundary conditions) that are smaller than $\lambda$ (see [39]) and $C$ is a constant.

**Proof.** By Theorem 7.1, the operators of $\{S_t\}$ corresponding to (9.19) satisfy all the requirements of smoothness imposed in Theorem 6.1. The operator $S'_t(z)$ is completely continuous by (2.19). (This formula holds also for the linear variational equation.) Hence, by Theorem 6.2, $\dim \mathfrak{A} \geq \dim E^u_\lambda$ where $E^u_\lambda$ is the subspace of unstable directions of $A'(z)$ at $z = z(\lambda)$. Since for (9.19) the operator

\[
\begin{align*}
\tag{9.22} 
A'(z)v &= \Delta v - f'(z)v + \lambda v \\
\end{align*}
\]

subject to the boundary conditions (9.19') or (9.19'') is self-adjoint, its spectrum is real and hence

\[
\tag{9.23} \dim E^u_\lambda = N(0, -A'(z)).
\]

To give a lower estimate for $\dim E^u_\lambda$, we use Courant's comparison principle. We have

\[
\tag{9.24} (-A'(z)v, v) \leq ((-\Delta + C - \lambda)v, v), \quad C \leq \max_{x \in \Omega} |f'(z(x, \lambda))|.
\]
By (9.20), we can choose \( C \) independently of \( \lambda \). From (9.24) according to Courant's principle we find that
\[
N(0, -A(z)) \geq N(0, -\Delta + C - \lambda) = N(\lambda - C, -\Delta).
\]
This together with (6.7) and (9.23) yields (9.21).

**Remark 9.7.** In §11 we shall establish an upper bound analogous to (9.6): \( \dim \mathcal{Y} \leq N(\lambda + C) \). Thus, \( N(\lambda + C) \geq \dim \mathcal{Y} \geq N(\lambda - C) \). Hence, bearing in mind the asymptotic expansion of \( N(\lambda) \) (see [39]):
\[
N(\lambda, -\Delta) = C_0\lambda^{n/2} + R(\lambda), \quad |R(\lambda)| \leq C_1\lambda^{(n-1)/2},
\]
we obtain
\[
\dim \mathcal{Y} = N(\lambda) + O(\lambda^{(n-1)/2}).
\]
To establish (9.25) it remains for us to construct stationary solutions \( z = z(\lambda), \lambda \in Y \), of the equation \( A(z) = 0 \) satisfying (9.20). It is particularly easy to find such solutions for \( n = 1 \).

**Proposition 9.2.** The equation in the domain \( \Omega = (0, 1) \subset \mathbb{R} \):
\[
(9.27) \quad \frac{d^2z}{dx^2} + f(z) + \lambda z - g(x) = 0, \quad z(0) = z(1) = 0, \quad g \in C([0, 1]),
\]
has a family of solutions \( z(x) = z(x, \lambda) \) satisfying (9.20) for any \( \lambda \in Y = R_0^0 \) where
\[
(9.28) \quad R_0^0 = \{ \lambda \geq 0: \text{dist}(\lambda, \{n^2n^2, n = 1, 2, \ldots\}) \geq b \}
\]
and \( b \) is sufficiently large.

**Proof.** The Green's function \( G \) of the operator \( d^2/dx^2 + \lambda \) on \( (0, 1) \) under zero boundary conditions is given by the formula
\[
G(x, y, \lambda) = \frac{1}{\sqrt{\lambda} \sin \sqrt{\lambda}} \left\{ \sin \left( \sqrt{\lambda} y \right) \sin \left( \sqrt{\lambda} (1 - x) \right), \quad 0 \leq y \leq x \leq 1. \right. \]
This implies that \( G = (\sqrt{\lambda} \sin \sqrt{\lambda})^{-1} K(x, y, \lambda) \), where \( |K| \leq 1 \). If \( \lambda \in R_0^0 \) and \( \lambda \geq 1 \), then \( \exists n \in \mathbb{N} : n^2n^2 + b \leq \lambda \leq (n + 1)^2n^2 - b \). Hence, \( |\sin \sqrt{\lambda}| \geq C\lambda^{-1/2}b \). Now (9.27) is equivalent to
\[
(9.29) \quad z = (\sqrt{\lambda} \sin \sqrt{\lambda})^{-1} (Kg + Kf(z)) = P(z)
\]
\[
\left( K\varphi = \int_0^1 K(x, y, \varphi(y)) \, dy \right).
\]
The operator \( P \) has the following two properties: 1) for any given \( M > 0 \) the ball \( B = \{ u : \| u \|_C \leq M \} \) is mapped by \( P \) into itself for any \( \lambda \in R_0^0 \) provided that \( b = b(M) \) is large enough (depending on \( M \)); 2) \( P \) is contracting on \( B \). For the factor \( (\sqrt{\lambda} \sin \sqrt{\lambda})^{-1} \) does not exceed \( C_1/b \) for \( \lambda \in R_0^0 \), the operator \( K \) is bounded in \( C \), and \( f(u) \) satisfies a Lipschitz condition. Choosing \( b \) large enough, we obtain 1) and 2). According to the theorem on contracting maps \( P \) has a fixed point \( z \) in \( B \), and this proves the assertion.

We now go over to the case \( n > 1 \) under the periodic boundary conditions (9.19'). We denote by \( \{H_n\} \) the scale of spaces generated by the operator
−Δ + a², a² > 0, under the periodic boundary conditions (9.19'). If λ does not belong to the spectrum of −Δ, then \((Δ + λ)\)^{-1}:H₄(T^n) → H₄(T^n) is a bounded operator. As is well known, H₄(T^n) for s > n/2 is closed under multiplication: uν ∈ H₄(T^n) if u, ν ∈ H₄(T^n). Moreover, if f ∈ C^q(R), s > n/2, and u ∈ H₄(T^n), then f(u) ∈ H₄(T^n).

We write

\[ R_0^α = \{λ > 0: \text{dist} (α, \{λ_κ\}) > bλ^{-α}\}, \]

where \{λ_κ\} is the spectrum of −Δ. We claim that for α > n/2 − 1 this set occupies a major part of the positive semi-axis. We denote by \(m_λ\) the number of eigenvalues of −Δ in the interval \(I_λ = [λ, λ + γλ^{1/2}]\) of length \(γλ^{1/2}\). The function \(N(λ)\) is known to satisfy (9.25). Obviously, \(m(λ) = N(λ + γλ^{1/2}) - N(λ)\). Using (9.25) we find that \(m_λ ≤ (C_2γ + C_3)λ^{(n-1)/2}\). The measure \(μ(I_λ \setminus R_0^α)\) of the set of points in \(I_λ\) whose distance from the spectrum is less than \(bλ^{-α}\) satisfies the inequality

\[ μ(I_λ \setminus R_0^α) ≤ 2m_λ bλ^{-α} ≤ 2λ^{n/2-1-α} (C_2γ + C_3). \]

Evidently, \(μ(I_λ \setminus R_0^α) / μ(I_λ) \leq \frac{2λ^{n/2-1-α} (C_2γ + C_3)}{γλ^{1/2}} = λ^{(n/2)-1-α} (2C_2 + 2C_3γ).\) If α > n/2 − 1, then this ratio tends to zero as λ → +∞.

**Proposition 9.3.** Let λ ∈ R_0^α. Then for any s ∈ R

\[ \| (Δ + λ)^{-1}ψ \|ₘ \leq b^{-1}\lambda^{-α} \| ψ \|ₘ, \quad ψ ∈ Hₘ. \]

The proof consists of a direct application of the definition of R_0^α and that of the norm in Hₘ.

**Theorem 9.5.** Let f(u) ∈ C^l, l > n/2 + 2, f(0) = f'(0) = 0, g(x) = g(x)λ^{(3-n)/2} for n ≥ 3, and g(x) = g_3(x) for n ≤ 3, where \(\|g_3(x)\|_{L^2} ≤ C\) and \(s' > 2 + (3/2)n\). Let λ ∈ R_0^α, where α = n/2 − 1 + δ, 0 < δ < 1/2. Then for sufficiently large λ there is a stationary solution \(u = u = u(λ)\) of (9.19):

\[ (9.30) \quad A(λ) = Δz + λz - f(z) - g(z) = 0 \]

subject to the boundary conditions (9.19''), for which

\[ (9.31) \quad \| z \|ₘ \leq Mλ^{(t-n)/2}, \quad M \text{ is independent of } λ. \]

**Proof.** To begin with we look for an approximate solution v_k of (9.30):

\[ A(v_k) = O(λ^{-k}) \]

in the form of a sum

\[ v = v_k = u_1λ^{-1} + u_2λ^{-2} + \ldots + u_kλ^{-k}, \quad \frac{n}{2} + \frac{1}{2} ≤ k ≤ \frac{n}{2} + 1. \]

Substituting this sum in (9.30) and comparing coefficients of equal powers of λ, we obtain in succession

\[ u_1 = g, \quad u_2 = −Δg, \quad u_3 = Δ^2g + \frac{1}{2}f' (0) g^2, \ldots \]
For $n \geq 3$ (the case $n < 3$ is simpler)

\begin{equation}
Av - h(x, \lambda), \quad \| h(x, \lambda) \|_s \leq C_1 \lambda^{(3-n)/2},
\end{equation}

\begin{equation}
\| v \|_s \leq C \lambda^{1-n)/2}, \quad s \geq \frac{n}{2}, \quad s \leq \frac{n}{2} + 1, \quad h(x, \lambda) \in H_s,
\end{equation}

by virtue of the conditions imposed on $g$ and $f(u)$.

We now look for the exact solution $z(x)$ of (9.30) in the form $z = v(x, \lambda) + w(x, \lambda)$. We have

\begin{equation}
\Delta w - (f(v + w) - f(v)) + \lambda w = -h(x, \lambda) \lambda^{-h} \equiv \psi(x, \lambda).
\end{equation}

For $\lambda \in R^+_\lambda$, the operator $\Delta + \lambda$ is invertible in $H_s$; consequently, (9.33) is equivalent to

\begin{equation}
w = (\Delta + \lambda)^{-1}(f(v + w) - f(v)) + (\Delta + \lambda)^{-1}\psi = P(w).
\end{equation}

We write $B_\lambda = \{w \in H_s : \| w \|_s \leq M \lambda^{(4-n)/2}\}$. We claim that $P(B_\lambda) \subset B_\lambda$ for sufficiently large $\lambda$. For as we indicated above, when $u \in H_s(T^n)$ and $s > n/2$, then $f(u) \in H_s(T^n)$ and $(\Delta + \lambda)^{-1} : H_s \rightarrow H_s$ ($\lambda \in R^+_\lambda$). By Proposition 9.3,

\begin{equation}
\| P(w) \|_s \leq C \lambda^\alpha (\| f(v + w) - f(v) \|_s + \| \psi \|_s).
\end{equation}

Since $s > n/2$ and $f(0) = f'(0) = 0$,

\begin{equation}
\| f(v + w) \|_s \leq C(M) \| v + w \|_s^2, \quad \| f(v) \|_s \leq C(M) \| v \|_s^2.
\end{equation}

Therefore, bearing in mind the definition of $\psi$ in (9.33), the estimates (9.32) for $v$ and $h$, and also the definition of $B_\lambda$, $w \in B_\lambda$, we obtain from (9.35)

\begin{equation}
\| P(w) \|_s \leq C \lambda^{\alpha} (\| v \|_s^2 + \| w \|_s^2 + \lambda^{-h} \| h \|_s) \leq C_2 \lambda^{-n+\alpha}.
\end{equation}

Since $\alpha = n/2 - 1 + \delta$, $\delta < 1/2$, we conclude from this that $P(w) \in B_\lambda$ for sufficiently large $\lambda$. We now prove that $P : B_\lambda \rightarrow B_\lambda$ is contracting:

\begin{equation}
\| P(w_1) - P(w_2) \|_s \leq C \lambda^\alpha \| f(v + w_1) - f(v + w_2) \|_s = \leq C \lambda^\alpha (\| v \|_s^2 + \| w_1 \|_s^2 + \| w_2 \|_s^2) \times \| w_1 - w_2 \|_s.
\end{equation}

Using the fact that $w_1, w_2 \in B_\lambda$ and $\| v \|_s$ satisfies (9.32), we find that

\begin{equation}
\| P(w_1) - P(w_2) \|_s \leq C \lambda^{3-n)/2} \| w_1 - w_2 \|_s.
\end{equation}

It follows that $P$ is contracting for sufficiently large $\lambda$. Therefore, (9.34) has a solution $w \in B_\lambda$ and this, in conjunction with the estimate (9.31) for $v$, proves the assertion of the theorem (the estimate (9.31) is for $z = v + w$).

Thus, we have proved the existence of a $z$ satisfying (9.20), consequently, the estimates (9.21) are proved also for $n > 1$.

Remark 9.8. Theorem 9.5 holds also under the zero boundary conditions (9.19') for $n \leq 3$. In this case we need to set $s = 2$. Then $H_s = W^2_2(\Omega) \cap \{u : \| u \|_2 = 0\}$. Clearly, this space is closed under
multiplication of its elements and under the operators \( u \rightarrow f(u) \) if \( f(0) = 0 \). Otherwise, the proof remains the same, with simplifications due to the fact that now \( k = 2 \). We mention that, as in the periodic case for \( n \leq 3 \) there is no need to require that \( g(x) \) is small: \( g(x) = g_1(x) \).

**Remark 9.9.** We consider the system of the type of chemical kinetics (9.9) under zero boundary conditions. We assume that \( \nu \) is fixed, \( \lambda \) is a large parameter, \( f(0) = 0, f'(0) = 0 \), and \( n = 2 \) or 3 (the case \( n = 1 \) is simpler). As in Theorem 9.5, we can establish the existence of stationary points \( z = z(x, \lambda) \) satisfying (9.31) for \( \lambda \) separated from the spectrum of the operator \( \nu \alpha \Delta \) by a distance \( b |\lambda|^{-\alpha} \), where \( \alpha = n/2 - 1 + \delta, 0 < \delta < 1/2 \); the function \( g = g(x) \) is the same as in Theorem 9.5. Then the lower estimate (9.18) holds for a fixed \( \nu \): \( \dim \mathfrak{A} \geq ch\sqrt{n/2} \).

The proof is based on (6.7). The lower bound for \( \dim E^\pm_\nu \) is obtained as follows. For simplicity we assume that the matrix \( \alpha \) is diagonalized: \( \{\mu_j, \rho_j\} = D, \text{Re} \mu_j > 0 \). The problem of evaluating \( \dim E^\pm_\nu \) then reduces to the determination of the number of eigenvalues \( \rho \), \( \text{Re} \rho > 0 \), of the following spectral problem:

\[
D \Delta w + \lambda w + C^{-1}f'(z)Cw = \rho w, \quad C^{-1}a_C D = D.
\]

Since \( z \) satisfies (9.31), \( f'(z) = O(z) \) and \( \lambda \) is separated from the spectrum by a distance \( b |\lambda|^{-\alpha} \), it is easy to show that the number of eigenvalues \( \rho \) with \( \text{Re} \rho > 0 \) is not less than \( C_1 \sqrt{n/2} \), that is, the same as for the principal part \( D \Delta w + \lambda w \) of (9.36). Thus, (9.18) holds for a fixed \( \nu \).

As was shown in Theorem 9.2, the upper estimate is the same.

**§10. Attractors of semigroups having a global Lyapunov function**

In this section we consider the attractors of semigroups having a global Lyapunov function. It turns out that the structure of the attractors of such semigroups can be described explicitly by means of the unstable invariant manifolds passing through fixed points.

**Definition 10.1.** Let \( \{S_t\} \) be a semigroup of operators, \( S_t: E \rightarrow E \). A point \( z \in E \) is said to be a fixed point of \( \{S_t\} \) if \( S_t z = z \) for all \( t \geq 0 \).

**Definition 10.2.** Let \( z \) be a fixed point of \( \{S_t\} \). The unstable invariant manifold emanating from \( z \) is the set \( M(z) \) of points \( u \in E \) with the following property: there is a continuous curve \( u(\tau) \in E, -\infty < \tau < +\infty \) such that 1) \( u(0) = u \); 2) \( S_t u(\tau) = u(\tau + t) \) for all \( \tau \in \mathbb{R} \), for all \( t > 0 \); 3) \( u(\tau) \rightarrow z \) as \( \tau \rightarrow -\infty \).

**Definition 10.3.** Let \( X \subset E \) be a weakly invariant set of a semigroup \( \{S_t\}: S_t X \subset X \) for all \( t \geq 0 \). A continuous functional \( \Phi \) on \( X \), \( \Phi: X \rightarrow \mathbb{R} \), is said to be a Lyapunov function of \( \{S_t\} \) on \( X \) if the following conditions are satisfied: 1) \( \Phi(S_t u) \) for any \( u \in X \), is a decreasing function of \( t \) for \( t \geq 0 \); 2) if \( \Phi(S_t u) = \Phi(S_0 u) \) for \( t > 0 \), then \( S_0 u \) is a fixed point of \( \{S_t\} \).
**Theorem 10.1.** Let $\mathcal{M}$ be a compact set that is invariant under a semigroup $\{S_t\}$, that is, $S_t\mathcal{M} = \mathcal{M}$ for all $t > 0$ and suppose that $\{S_t\}$ has a Lyapunov function $\Phi$ on $\mathcal{M}$. We assume that the set $\mathcal{M}$ of fixed points of $\{S_t\}$ is finite. We assume further that $S_t u$ for any $u \in \mathcal{M}$ depends continuously on $t$ in $E$. Then

$$\mathcal{M} \subset \bigcup_{z \in \mathcal{M}} M(z),$$

where $M(z)$ is the unstable invariant manifold emanating from $z$.

**Proof.** Let $u_0 \in \mathcal{M}$. Since $S_t \mathcal{M} = \mathcal{M}$, there is a curve $u(\tau) \in \mathcal{M}$, $\tau \in \mathbb{R}$, such that $u(0) = u_0$ and $S_t u(\tau) = u(t + \tau)$ for all $t \geq 0$, for all $\tau \in \mathbb{R}$. For if $\tau > 0$, we set $u(\tau) = S_\tau u_0$. Since $S_\tau \mathcal{M} = \mathcal{M}$, there is a $u_1 \in \mathcal{M}$ such that $S_{\tau} u_1 = u_0$. We set $u(-1) = u_1$. For $-1 \leq \tau \leq 0$ we set $u(\tau) = S_{\tau + 1} u_1$. Similarly we define $u(\tau)$ for $-k - 1 \leq \tau \leq -k$ ($k = 1, 2, \ldots$). It is easy to see that the thus constructed curve $u(\tau)$ has all the required properties and is, in addition, continuous.

We now claim that, $u(\tau) \to z$ as $\tau \to -\infty$, where $z$ is a fixed point of $\{S_t\}$. We set

$$X_0^t = \bigcup_{\tau \leq t} u(\tau), \quad X_\tau = \overline{X_0^\tau}.$$  

Since $\mathcal{M}$ is compact, so is $X_\tau \subset \mathcal{M}$. Obviously, $X_t \subset X_\tau$ for $t \leq \tau$. We set

$$X = \bigcap_{t < 0} X_t.$$  

Since the $X_t$ are compact, $X \neq \emptyset$.

We now show that $X$ consists of fixed points. First of all, we note that $\tau \to \Phi(u(\tau))$ is a monotonically decreasing function. Since $u(\tau) \in \mathcal{M}$ for all $\tau$ and $\mathcal{M}$ is compact, $\psi(u(\tau))$ is bounded and therefore, the limit

$$\varphi_- = \lim_{\tau \to -\infty} \Phi(u(\tau))$$

exists. We now observe that

$$\Phi(x) = \varphi_- \quad \forall x \in X.$$  

For if $x \in X$, then by (10.3) and (10.2) there is a sequence $\tau_k \to -\infty$ such that $u(\tau_k) \to x$. Since the functional $\Phi$ is continuous, $\Phi(u(\tau_k)) \to \Phi(x)$ and as a consequence of (10.4), $\Phi(x) = \varphi_-$. We now claim that

$$S_t v \in X \quad \forall v \in X.$$  

For if $v \in X$, then there are $v_k \in X_0^t$ such that $v_k \to v$ as $k \to -\infty$. We set $u = S_t v$. We write $u_k = S_t v_k$. It is obvious that $u_k \in X_0^t$ and $\theta_k = t_k + t \to -\infty$ as $k \to +\infty$. Since $S_t$ is continuous, $u_k = S_t v_k \to S_t v = u$ as $k \to +\infty$. Hence, $u \in X$, which proves (10.6).

We now show that if $v \in X$, then $v$ is a fixed point. For if $v \in X$, then by (10.6) $S_t v \in X$ for $t > 0$. As a consequence of (10.5), $\Phi(S_0 v) = \Phi(S_t v) = \varphi_-$ and, by Definition 10.3, 2), $S_0 v$ is a fixed point.
Next we show that
\[(10.7) \quad \text{dist}(u(\tau), X) \to 0 \quad \text{as} \quad \tau \to -\infty.\]

For otherwise, there would be a sequence \(\tau_k \to -\infty\) such that \(\text{dist}(u(\tau_k), X) \geq \epsilon > 0\). Since \(X\) is compact, by going to a subsequence we may assume that \(u(\tau_k) \to u \in \mathfrak{A}\). Evidently, \(u \in X\), which contradicts the assumption that \(\epsilon\) is positive.

We now claim that \(X\) is connected (see [5]). Suppose the contrary, that is \(X = X_1 \cup X_2\), where \(X_1 \cap X_2 = \emptyset\), \(X_1 \neq \emptyset\), \(X_2 \neq \emptyset\) and there are open neighbourhoods \(\Omega_1 \supset X_1\) and \(\Omega_2 \supset X_2\) such that \(\Omega_1 \cap \Omega_2 = \emptyset\). We set \(\Omega = \Omega_1 \cup \Omega_2\). By (10.7), \(u(\tau) \in \Omega\) for \(\tau \in J = (-\infty, -\tau_0)\) and a sufficiently large \(\tau_0\). Since the map \(u : J \to \Omega\) defined by the correspondence \(\tau \to u(\tau)\) is continuous, the inverse images \(J_1 = u^{-1}(\Omega_1)\) and \(J_2 = u^{-1}(\Omega_2)\) are open, and it is obvious that \(J = J_1 \cup J_2\). Since \(J\) is connected, one of the sets \(J_1\) and \(J_2\), say \(J_1\), must be empty. Now we find from the definition of \(X\) that \(X_1\) is empty, and this is a contradiction. Thus, \(X\) is connected. Since \(X \subset \mathfrak{R}\) and \(\mathfrak{R}\) is finite, \(X\) is also finite and therefore consists of a single point \(z \in \mathfrak{R}\). It follows from (10.7) that \(u(\tau) \to z\) as \(\tau \to -\infty\), and the curve \(u(\tau)\) satisfies all the requirements of Definition 10.2. Therefore, \(u(0) = u_0 \in M(z)\).

**Theorem 10.2.** Suppose that a semigroup \(\{S_t\}\) has a maximal attractor \(\mathfrak{A}\) and that all the hypotheses of Theorem 10.1 are satisfied. Then
\[(10.8) \quad \mathfrak{A} = \bigcup_{z \in \mathfrak{R}} M(z),\]
where \(\mathfrak{R}\) is the set of fixed points of \(\{S_t\}\).

**Proof.** (10.1) holds by Theorem 10.1. It remains to prove the reverse inclusion, that is, \(M(z) \subset \mathfrak{A} \forall z \in \mathfrak{R}\).

Let \(x \in M(z)\) and let \(x(\tau)\) be a curve satisfying the conditions 1)-3) of Definition 10.2. The set \(B = \cup x(\tau), \tau \in \mathbb{R}\), is bounded in \(E\). For \(u(\tau) \to z\) as \(\tau \to -\infty\), therefore, the \(x(\tau)\) are bounded for \(\tau < -K_+\). As \(\tau \to +\infty\), \(\text{dist}(x(\tau), \mathfrak{A}) \to 0\) by the attracting property of a maximal attractor. Since \(\mathfrak{A}\) is bounded, so are the \(x(\tau), \tau \geq K_+\). Since \(x(\tau)\) is continuous, it is bounded for \(\tau \in [-K_-, K_+]\). Hence, \(B\) is bounded.

By the attracting property of a maximal attractor, \(\text{dist}(S_tB, \mathfrak{A}) \to 0\) as \(t \to +\infty\). It follows from Definition 10.2, 2) that \(B\) is invariant: \(S_tB = B\) for all \(t > 0\). Hence \(\text{dist}(B, \mathfrak{A}) = \lim_{t \to +\infty} \text{dist}(S_tB, \mathfrak{A}) = 0\). Since \(\mathfrak{A}\) is closed, this implies that \(B \subset \mathfrak{A}\) hence \(z \in \mathfrak{A}\). Consequently, \(M(z) \subset \mathfrak{A}\).

We now give an example of an equation whose attractor has the form (10.8).

We consider on the torus \(T^n\) an equation of the form (2.1) in which
\[(10.9) \quad a_i(\xi) = \partial a(\xi)/\partial \xi_i, \quad b_i \equiv 0.\]
We assume that (2.3) with \(p_1 = 2\) and the strong ellipticity condition (2.2) hold, and that \(|a_{ij}(\xi)| \leq C\) for all \(\xi \in \mathbb{R}^n, a_{ij} = \partial^2 a/\partial \xi_i \partial \xi_j\).
Furthermore, the conditions (2.4), (2.5), and (2.6) hold with for all $p_0$, $n \leq 4$ and $p_0 \leq 1 + n/(n - 4)$, $n > 4$. The right-hand side of this equation (2.1) is the functional derivative $-\delta \Phi/\delta u(x)$ of the functional

\begin{equation}
\Phi(u) = \int_{\Omega} \left( a(\nabla u) + F(x, u) - \frac{1}{2} \lambda u^2 \right) dx, \quad F(x, u) = \int_0^u f(u, \zeta) d\zeta,
\end{equation}

which, as will be shown below, is the global Lyapunov function of this equation. Thus, we consider the equation

\begin{equation}
\partial_t u = -\frac{\delta \Phi(u)}{\delta u(x)} = \sum_{i=1}^n \partial_i(a_i(\nabla u)) - f(x, u) + \lambda u, \quad u|_{\partial \Omega} = 0.
\end{equation}

We remark that we restrict ourselves to the case of an equation (10.11) only for the sake of simplicity. Broad generalizations are possible.

**Theorem 10.3.** Equation (10.11) generates a semigroup $\{S_t\}$, $S_t : H \to H$ ($H = L_2(\Omega)$). This semigroup has an attractor $\mathcal{A}$ that is bounded in $H^2(\Omega)$. The functional $\Phi$ defined by (10.10) is continuous on $\mathfrak{A}$ and is the Lyapunov function for $\{S_t\}$ on $\mathfrak{A}$. If

\begin{equation}
A(u) = \sum_{i=1}^n \partial_i(a_i(\nabla u)) - f(x, u) + \lambda u = 0
\end{equation}

has in $H_2(\Omega)$ a finite set $\mathfrak{A}$ of solutions $u = z_j(x)$ ($j = 1, \ldots, k$) then $\mathfrak{A}$ is the union of the unstable invariant manifolds $M(z_j)$ ($j = 1, \ldots, k$) emanating from $z_j$, that is, (10.8) holds.

**Proof.** According to Theorem 2.3, the semigroup $\{S_t\}$ corresponding to (10.11) is an attractor of $\mathcal{A}$ in $H^2(\Omega)$. We now claim that $\mathfrak{A}$ is bounded in $H^2(\Omega)$. To begin with, we derive a formal estimate.

Multiplying (10.11) by $\partial_t u$ and integrating with respect to $x$ and $t$, we obtain

\begin{equation}
\int_0^t ||\partial_t u||^2 dt + \int_{\Omega} a(\nabla u) dx + \int_{\Omega} F(x, u) dx - \frac{1}{2} \int_0^t ||u||^2 dt = 0.
\end{equation}

Since $u(0) \in \mathfrak{A} \subset H_1$, all the terms in (10.13) are bounded in modulus by a constant. We differentiate (10.11) with respect to $t$, multiply by $t^2 \partial_t u$, and integrate with respect to $x$ and $t$. As a result,

\begin{equation}
||t \partial_t u||^2 + \mu_0 \int_0^t t^2 ||\partial_t u||^2 dt \leq C \int_0^t t ||\partial_t u||^2 dt + \int_0^t \lambda t^2 ||\partial_t u||^2 dt.
\end{equation}

By (10.13), the right-hand side of (10.14) is bounded by a constant depending only on $t$. Setting $t = 1$, we conclude from (10.14) that $\partial_t u(1)$ is bounded in $L_2(\Omega)$. By (10.11), since $\partial_t u(1)$ is bounded in $L_2(\Omega)$ and $u(1)$ is bounded in $H_4(\Omega)$, it follows that $u(1)$ is bounded in $H_6(\Omega)$. These calculations can be repeated verbatim for the Galerkin approximations $u^N$ of (10.11), and all the differentiations are justified. This proves that $u^N(1)$ and therefore $u(1)$ are bounded in $H_2(\Omega)$ by a constant depending only on $\mathfrak{A}$. Observing that $\mathfrak{A} = S_1\mathfrak{A}$ we deduce that the attractor $\mathfrak{A}$ is bounded in $H_2(\Omega)$. 

**References**
We now verify that $\Phi$ is continuous on $\mathfrak{F}$:

\[ |\Phi(u_1) - \Phi(u_2)| \leq \left| \int_{\mathbb{T}^n} \left( (a(\nabla u_1) - a(\nabla u_2)) + (F(x, u_1) - F(x, u_2)) \right) dx \right| + 
\]

\[ + \frac{1}{2} \lambda \left| u_1 \right|^2 - \left| u_2 \right|^2 \leq \left| \sum_{i=1}^{n} \int_{\mathbb{T}^n} a \left( \partial_i \nabla u_1 + (1 - \theta) \nabla u_2 \right) d\theta \partial_i (u_1 - u_2) + 
\]

\[ + \int_{\mathbb{T}^n} f(x, \theta u_1 + (1 - \theta) u_2) (u_1 - u_2) d\theta dx + \frac{1}{2} \lambda \left| u_1 \right|^2 - \left| u_2 \right|^2 \right|. \]

Integrating by parts, we interchange $\partial_i$ with $a_i$, and using the conditions imposed on $a$ and $f$, we obtain

\[ |\Phi(u_1) - \Phi(u_2)| \leq C (\left| u_1 \right|_2 + \left| u_2 \right|_2 + 1) \left| u_1 - u_2 \right| + 
\]

\[ + C \int_{\mathbb{T}^n} \left( |u_1|^{n/(n-4)} + |u_2|^{n/(n-4)} + 1 \right) \left| u_1 - u_2 \right| dx \leq 
\]

\[ \leq C_1 (\left| u_1 \right|_2 + \left| u_2 \right|_2 + \left| u_1 \right|^{n/(n-4)} + |u_2|^{n/(n-4)} + 1) \left| u_1 - u_2 \right|. \]

We now prove that

\[(10.15) \int_{\mathbb{T}^n} |A(u(s))|^2 ds = \Phi(u(t)) - \Phi(u(\tau)), \quad 0 \leq t \leq \tau, \]

where $A(u)$ is given by (10.12), $u(\tau) \in \mathfrak{F}$. It is easy to establish by means of (10.13) and (10.14) that $\Phi(u(t))$ is continuously differentiable in $t$ for all $t \geq 0$. (We use the extension of $u(t)$ on $\mathfrak{F}$ to negative $t$.) The derivative of $\Phi(u(t))$ is

\[
\frac{d}{dt} \Phi(u(t)) = \int_{\mathbb{T}^n} \frac{\delta \Phi}{\delta u(x)} \frac{\partial u}{\partial t} dx = -\int_{\mathbb{T}^n} A(u(t)) \frac{\partial u}{\partial t} dx = \int_{\mathbb{T}^n} |A(u(t))|^2 dx.
\]

(In the derivation we have used the fact that $u(t)$ is the solution of (10.11).) Integrating this equality, we obtain (10.15). The left-hand side of (10.15) vanishes if $u(t)$ is the solution of (10.12) for all $t$. It is obvious that the set of solutions of (10.12) is identical with that of fixed points of $\{S_t\}$. Therefore, $\Phi$ is a Lyapunov function of the semigroup $\{S_t\}$, and if the set $\mathfrak{M}$ of solutions of (10.12) is finite and consists of functions $z_i(x)$ ($i = 1, \ldots, \kappa$), then the attractor $\mathfrak{A}$ has the representation (10.8).

§11. Regular attractors of semigroups having a Lyapunov function

Here we consider semigroups $\{S_t\}$, $S_t : E \to E$, having a Lyapunov function (see Definition 10.3). We assume that the operators $S_t$ are Fréchet differentiable. By Theorem 10.2, the maximal attractors of such semigroups have the form (10.8). When the operator $S_t(u)$ satisfies certain conditions and the fixed points $z_i$ are hyperbolic (see Definition 11.1), then the attractor $\mathfrak{A}$ has some additional properties. Such attractors are said to be regular. Detailed proofs of the theorems stated in this section are in [26].
Definition 11.1. A fixed point \( z \) of a map \( S \) is said to be hyperbolic if 1) in a neighbourhood \( O \) of \( z \) the operator \( S \) has for \( u \in O \) the Fréchet differential \( S'(u) : E \to E; \) 2) the linear operator \( S'(u) \) satisfies the Hölder condition (6.3); 3) the spectrum of \( S'(z) \) contains no points \( \lambda \) on the unit circle \( |\lambda| = 1. \) A fixed point \( z \) of a semigroup \( \{S_t\} \) is said to be hyperbolic if for every \( t > 0 \) the map \( S_t \) is hyperbolic at \( z \) and furthermore: 4) the invariant subspaces \( E_+ \) and \( E_- \) corresponding to the parts \( \sigma_+ \) and \( \sigma_- \) of the spectrum of \( S_t'(z) \) that are located, respectively, in the domains \( |\lambda| > 1 \) and \( |\lambda| < 1 \) are independent of \( t, \) and \( E_+ \) is finite-dimensional.

If a semigroup \( \{S_t\} \) has a Lyapunov function and \( z \) is a hyperbolic fixed point of it, then there are numbers \( c > 0 \) and \( r = 1 + \epsilon \) (\( \epsilon > 0 \) sufficiently small) such that

\[
M(z, O_r(z), S_t, r) \subseteq M(z) \cap O_r(z) \subseteq M(z, O_r(z), S_t, r).
\]

Therefore, the manifold \( M(z) \) in a neighbourhood of \( z \) is described by (6.4), that is, its dimension is \( n = \dim E_+ \). The following theorem holds:

**Theorem 11.1.** Suppose that \( \{S_t\} \) has the following properties: 1) the map \( E \times [0, +\infty) \to E \) defined by \( (u, t) \to S_t u \) is continuous; 2) \( \{S_t\} \) has a global Lyapunov function on \( E. \)

Let \( z \) be a hyperbolic fixed point of \( \{S_t\} \) such that 3) for each \( t > 0 \) the restriction of \( S_t \) to \( M(z) \) is a one-to-one map onto \( M(z); \) 4) the inverse map \( S_{t}^{-1} \) is continuous on \( M(z); \) 5) the linear operators \( S'_t(u) \) for \( u \in M(z) \) and \( t \geq 0 \) have zero kernel.

Then \( M(z) \) is a finite-dimensional \( C^1 \)-submanifold of \( E \) that is diffeomorphic to \( \mathbb{R}^n \), where \( n = \dim E_+ \).

**Theorem 11.2.** Suppose that the set \( \mathcal{N} \) of fixed points of \( \{S_t\} \) is finite. Suppose also that \( \{S_t\} \) and all its fixed points satisfy the hypotheses of Theorem 11.1 and the following additional conditions: 1) for each \( t > 0 \) the operator \( S_t \) maps bounded sets in \( E \) to precompact sets in \( E; \) 2) the semigroup \( \{S_t\} \) is uniformly bounded in \( E \) (see §1).

Then the set \( (\mathcal{N}) \)

\[
\mathcal{N} = \bigcup_{z \in \mathcal{N}} M(z)
\]

is a maximal attractor of \( \{S_t\} \) and the unstable invariant manifolds \( M(z_j), z_j \in \mathcal{N}, \) are \( C^1 \)-submanifolds of \( E \) and diffeomorphic to \( \mathbb{R}^{n_j} \), where \( n_j = \dim E_+(z_j). \) Suppose next that the points \( z_j \in \mathcal{N} \) are numbered so that

\[
\Phi(z_1) \leq \Phi(z_2) \leq \ldots \leq \Phi(z_\kappa),
\]

where \( \Phi \) is the Lyapunov function and \( \kappa \) is the number of fixed points.

Sets \( \mathcal{M}_k \) are defined as follows:

\[
\mathcal{M}_k = \bigcup_{j=1}^{k} M(z_j) \quad (k = 1, \ldots, \kappa), \quad \mathcal{M}_0 = \emptyset.
\]

(1) For the definition of \( M(z), \) see §10.
Then the following assertions hold for \( j = 1, \ldots, \kappa \): 
\[
\mathcal{M}_j \text{ is compact in } E; \quad S_t \mathcal{M}_j = \mathcal{M}_j \text{ for all } t \geq 0; \quad \mathcal{M}_j \text{ is stable}\(^1\) in } E \text{ with respect to } \{S_t\}; \quad \overline{M(z_j) \setminus M(z_j)} = \partial M(z_j) \subset \mathcal{M}_{j-1}; \quad S_t \partial M(z_j) = \partial M(z_j) \text{ for all } t \geq 0; \quad \text{and } \lim_{t \to \infty} \text{dist} (S_t K, \mathcal{M}_{j-1}) = 0 \text{ for any compact set } K \subset \mathcal{M}_j \setminus z_j.
\]

A set \( \mathcal{H} \) is called a regular attractor if it satisfies all the requirements stated in Theorem 11.2 for some numbering of the points \( z_1, \ldots, z_\kappa \).

**Remark 11.1.** In contrast to Theorems 1.1 and 1.2, the hypotheses of Theorem 11.2 do not include the existence of an absorbing set. In Theorem 11.2 the condition 1), which requires the compactness of \( S_t, t > 0 \), can also be relaxed (see Theorem 11.5). This enables us to prove the existence of regular attractors for hyperbolic equations (see [26], [38], and Example 11.2).

**Example 11.1.** We consider the following parabolic equation, which is a special case of Example 2.2:

\[
\partial_t u = a(x, u)(\Delta u - f(x, u) + \lambda u - g(x)), \quad u|_{\partial \Omega} = 0,
\]

where \( a(x, u) > 0, g \in E_0, E_0 = C^\alpha(\Omega) \cap \{g: g|_{\partial \Omega} = 0\}, \) and \( \alpha > 0 \). We assume that for each \( \lambda > 0 \) the two conditions in Example 2.2 are satisfied. We set

\[
A_0(u) = \Delta u - f(x, u) + \lambda u, \quad A'_0(u) v = \Delta v - f'(x, u) v + \lambda v.
\]

It is obvious that \( A_0: E \to E_0 \) and \( A'_0(u): E \to E_0 \), where \( E \) is defined by (2.16). We call a function \( g(x), g \in E_0 \), a regular value of the operator \( A_0 \) if the equation \( A_0(u) = g \) has finitely many solutions \( z_1, \ldots, z_\kappa \) and \( A'_0(z_j): E \to E_0 \) for each \( z_j \) is an invertible operator.

**Theorem 11.3.** Let \( g \in E_0 \) be a regular value of \( A_0 \) defined by (11.3). Then the semigroup \( \{S_t\}, S_t: E \to E \), corresponding to (11.2) has a regular attractor. Its Hausdorff dimension is

\[
\dim \mathcal{H} = \max \dim E^u_j(z_j),
\]

where \( E^u_j(z_j) \) is the subspace of unstable directions corresponding to \( A'_0(z_j) \).

The proof (see [26]) is based on an application of Theorem 11.2 in which the functional

\[
\Phi(u) = \int_0^1 \left( \frac{1}{2} |\nabla u|^2 + F(x, u) - \frac{\lambda}{2} u^2 + gu \right) dx,
\]

\[
F(x, u) = \int_0^u f(x, \zeta) d\zeta
\]

is taken for the Lyapunov function. We now give an upper bound for the dimension of the attractor of (11.2).

\(^1\) A set \( X \) is said to be stable with respect to \( \{S_t\} \) if \( \forall \varepsilon > 0 \exists \delta > 0 \) such that \( S_t X \) for any \( x \) in a \( \delta \)-neighbourhood of \( X \) belongs to an \( \varepsilon \)-neighbourhood of \( X \) for all \( t \geq 0 \).
Theorem 11.4. Suppose that the hypotheses of Theorem 11.3 are satisfied and that (9.4) holds. Then

\[(11.6) \quad \dim X \leq N(\lambda + C),\]

where \(N(\lambda)\) is the same as in (9.21) and \(C\) is independent of \(\lambda\).

Proof. By (11.4) it is sufficient to find an upper bound for \(\dim E_+^0(z_1)\). Clearly,

\[(A_0'(z_j) v, v) = (\Delta v - f'(x, z_j) v + \lambda v, v) \leq (\Delta v, v) + (\lambda + C)(v, v) = ((\Delta + \lambda I + CI) v, v),\]

where (,) is the scalar product in \(L_2(\Omega)\) and \(C\) is independent of \(\lambda\) by (2.14); \(f'(u) \geq -f^0\).

Using Courant's comparison principle we find that the number of positive eigenvalues of \(A_0'(z_j)\) is not greater than that of the operator \(\Delta + \lambda I + CI\), that is, \(\dim E_+^0(z_j) \leq N(\lambda + C)\). Since the eigenvectors of \(A_0'(z_j)\) in \(L_2(\Omega)\) and in \(E\) are the same, this implies (11.6).

Remark 11.2. The condition on \(g\) requiring that \(g\) should be a regular value of \(A_0\) is the condition of general position. (This follows from the Sard-Smale theorem.) It is also easy to show that a \(g\) in general position is a regular value of \(A_0 = A_0(\lambda)\) for \(\lambda\) from an everywhere dense set on the positive semi-axis, that is, (11.6) holds as \(\lambda \to +\infty\) for \(\lambda\) chosen from such a set. Taking into account (9.25) we find that (9.26) holds in the case of general position for \(n \leq 3\), that is, \(\dim \mathcal{A} = N(\lambda) + O(\lambda^{(n+1)/2})\) on some unbounded set of \(\lambda > 0\).

Example 11.2. In a domain \(\Omega \subset \mathbb{R}^3\) we consider the hyperbolic equation

\[(11.7) \quad \partial_t^2 u + e \partial_t u = \Delta u - f(u) - g(x) + \lambda u, \quad u|_{t=0} = 0,\]

where \(e > 0\) and \(g \in H = L_2(\Omega)\).
We assume that \( f(u) \in C^3(\mathbb{R}) \) and

\[
\begin{align*}
(f(u) - \lambda u, u) & \geq -C_\lambda, \quad f'(u) \geq -C \quad \forall u \in \mathbb{R}; \\
|f'(u)| & \leq C(|u|^{2-\gamma} + 1), \quad \gamma > 0; \\
|f''(u)| & \leq C_1(1 + |u|), \quad |f''(u)| \leq C_1 \quad \forall u \in \mathbb{R}.
\end{align*}
\]

Similar conditions are imposed in the cases \( n < 3 \) and \( n > 3 \). We introduce the following Hilbert spaces:

\[
E = \{ y = (u, p): u \in H_1 \cap \{ u |_{\Omega_0} = 0 \}, \quad p \in H \}, \\
E_1 = \{ y = (u, p): u \in H_2 \cap \{ u |_{\Omega_0} = 0 \}, \quad p \in H_1 \cap \{ p |_{\Omega_0} = 0 \} \}.
\]

As is known, the correspondence \((u_0, p_0) \rightarrow (u(t), \partial_t u(t))\), where \( u(t) = u(x, t) \), is a solution of (11.7) subject to the initial condition \( u(0) = u_0 \), \( \partial_t u(0) = p_0 \), defines a semigroup \( \{ S_t \} \), \( S_t: E \rightarrow E, S_t: E_1 \rightarrow E_1 \).

**Theorem 11.6.** Let \( g \in H \) be a regular value of the operator \( A_0 \) defined by (11.3), \( A_0: H_2 \cap \{ u |_{\Omega_0} = 0 \} \rightarrow H \). Then the semigroup \( \{ S_t \} \) corresponding to (11.7) has a regular \( (E_1, E) \)-attractor \( A \), and its Hausdorff dimension has the upper bound (11.6) and the lower bound (9.21) if \( f(0) = f'(0) = 0 \).

The proof of Theorem 11.6 is based on Theorem 11.5. The Lyapunov function is \( \Phi(u) \) defined by (11.5).

**References**


Translated by D. Mathon

Moscow Transport Engineering Institute
Moscow State University

Received by the Editors 23 December 1982